Symmetric Functions and the Symmetric Group S_n

A thesis submitted for the Degree of PhD in Physics

M A Salam

University of Canterbury, Christchurch, New Zealand.
Te Whare Waananga O Waitaha, Otautahi, Aotearoa.

April 1991

To my father

Contents

Abstract

A simple method for the embedding $O_n \rightarrow S_n$ of ordinary and spin irreps in both n-dependent notation and an n-independent reduced notation is given. Basic spin irreps and ordinary irreps are combined using the properties of Q-functions and raising operators in order to give a complete set of branching rules of $O_n \rightarrow S_n$ for spin irreps. The modification rules for Q-functions given by Morris are redefined to yield a complete and unambiguous set of rules.

Properties of shifted tableaux have been explored in order to improve the algorithm for the calculation of Q-function outer products. A simple technique has been established for finding out the highest and lowest partitions in the expansion of Q-function outer products. Using these techniques and Young's raising operators, the Kronecker product for S_n spin irreps has been completed.

A number of properties of Young's raising operator as applied to S-functions and Schur's Q-functions are noted. The order of evaluating the action of inverse raising operators is found to require careful specification and the maximum power of the operators δ_{ij} is determined. The operation of inverse raising operator on a partition λ is found to be the same as for its conjugate $\tilde{\lambda}$. A new definition of Shifted Lattice Property that can efficiently remove all the dead tableaux in the Q-function analogue of the Littlewood-Richardson rule is introduced. A simple combinatorial analogue of raising and inverse raising operators is given.

The $q-$deformation of symmetric functions is introduced leading to $q-$analogues of many well-known relationships in the theory of symmetric functions. A $q-$analogue of the spin and ordinary characters of S_n is given by making use of a method that closely parallels that of quantum groups. This formalism leads to a very simple technique for the construction of twisted and untwisted q-vertex operators. An isomorphism between the space of $q-$vertex operators and the ring of $q-$deformed Hall-Littlewood symmetric functions has been found.

Chapter 1

Introduction

Mathematics and particularly group theory can be used to solve specific problems arising in physics, and to provide the very language in which the laws of physics may be formulated. The very assertion of symmetry is often the most profound formulation of a physical law or the key step in the development of a new theory. There are many simple and complex symmetries found in nature. One of the most widely used ones in atomic and nuclear physics is permutation symmetry. Group theory is the most profound mathematical tool for the quantitative and analytical description of these symmetries.

The theory of symmetric functions plays a very important role in such formulations, but unfortunately has been much neglected by physicists. In very recent work on the symmetric functions [19, 28], mostly done by mathematicians, it has been shown how beautifully a purely mathematical theory can be reconciled with the problems of physics such as the quantum Hall effect, solitons, string theory, etc. The main purpose of this thesis is to explore these new directions and develop formalisms of the theory of symmetric functions relevant to these physical problems and study some of the applications.

The Schur symmetric function, or simply the S-function, is a special kind of symmetric function. The S-function was known to mathematicians since 1841 but it was Schur who made explicit the relationship between S-functions and the characters of the symmetric group [26]. In 1911, in a remarkable paper [1] Schur introduced another type of symmetric function, known as the Schur Q-function in the description of the projective representations of the symmetric group. These symmetric functions were further studied and developed by Hall, Littlewood and Macdonald [2, 11, 18, 20], and are commonly known as Hall-Littlewood symmetric functions. The theory of symmetric functions has become an integral part of group theory. The Hall-Littlewood symmetric functions and their special cases find many applications in physics. The following three main areas of physics are of particular interest.

- Soliton physics,
- Quantum Hall effect,
- Vertex operators.

1.1 Soliton physics

Solitons are nonlinear localized waves which

1. propagate without changing their properties (shape, velocity, etc.), and
2. are stable against mutual collisions in which each wave conserves its identity.

The solitary waves were first observed in a canal in August 1834, by a Scottish scientist and engineer named John Scott-Russell [27]. He proposed that the stability of the wave he had observed resulted from intrinsic properties of the wave's motion rather than from the circumstances of its generation. This view was not immediately accepted. In 1895 however, D. J. Korteweg and Hendrik de Vries gave a complete analytic treatment of a nonlinear equation in hydrodynamics and showed that localized nondissipative waves could exist. Since then the soliton has become a well-recognized phenomenon in various fields of physics; fluid dynamics, plasma physics, nonlinear optics, solid state physics, low-temperature physics, elementary particle physics, astrophysics, biophysics. Study of the soliton in all these areas of physics is called *soliton physics* and is distributed through a large amount of literature. The name *soliton* was given because of its particle-like behaviour, although in elementary particle physics it is sometimes regarded as a field structure localized in space and time.

The equation derived by Korteweg and De Vries is generally known as KdV equation and can be written as

$$u_t + (c + \alpha u)u_x + \beta u_{xxx} = 0, \tag{1.1}$$

where $c = (gh)^{1/2}$, $\alpha = \frac{3}{2}(c/h)$, $\beta = \frac{1}{6}c(h^2 - 3T\rho g)$, h is the average depth of the canal, u the elevation of the surface, g the gravitational constant, T the surface tension and ρ the density.

A more useful form of the KdV equation is;

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.2}$$

where the second term is a nonlinear term which acts to steepen the wave, whereas the third term is a dispersion term which spreads out the wave. The balance between these opposing effects is the origin of the constant wave form and explains the existence of solitary waves.

1.1.1 The KP equation and the S-functions

There are many soliton equations but the most widely studied one in $(2+1)$ dimensions is the Kadometsev-Petviashvili (KP) equation [29],

$$u_t u_x + 6uu_{xx} + u_{xxxx} + 3u_{yy} = 0. \tag{1.3}$$

In the KP equation, resonant interaction between solitons takes place and subsequently the spatial structure where three solitons emerge from a point in space appears. The algebraic properties of the KP equation is explored by the τ function approach [30, 31]. Usually the KP equation is transformed to Hirota form by the change of variable,

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau.$$

Then we have

$$(D_x^4 + D_x D_t + 3D_y^2)\tau\cdot\tau = 0,$$

where the Hirota derivatives are defined by

$$D_x^m D_t^n \tau\cdot\tau \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \tau(x,t)\tau(x',t')\,|_{x'=x,t'=t}.$$

The partition notation of the derivatives and the Hirota derivatives is more useful.

$$\partial_\lambda \equiv \frac{\partial^l}{\partial x_{\lambda_1}\cdots\partial x_{\lambda_l}},$$
$$D_\lambda \equiv D_{x_{\lambda_1}}\cdots D_{x_{\lambda_l}},$$

for any partition $\lambda \equiv (\lambda_1,\cdots,\lambda_l)$. In this notation the KP equation has the Hirota form

$$(D_{1^4} + 3D_{2^2} - 4D_{31})\tau\cdot\tau = 0, \tag{1.4}$$

where τ is the function of the sequence of variables $\boldsymbol{x} = (x_1, x_2, x_3, \cdots)$.

If $\varphi_1,\cdots,\varphi_N$ are any functions of $\boldsymbol{x}$ satisfying

$$\partial_j\varphi_i = \partial_1^j\varphi_i, \tag{1.5}$$

for $i = 1,\cdots,N$ and $j\in\mathcal{Z}^+$, and $\partial_k = \frac{\partial}{\partial x_k}$, then the Wronskian determinant

$$W(\varphi_1,\cdots,\varphi_N) = \det\left(\partial^{i-1}\varphi_j\right),$$

with $\partial \equiv \frac{\partial}{\partial x_1}$ satisfies (1.4). The polynomial functions satisfying (1.5) are obtained from the following generating function;

$$\prod_{i=1}^{n}(1-\alpha_i z)^{-1} = \exp\left(\sum_{j=1}^{\infty} x_j z^j\right) = \sum_{k=0}^{\infty} h_k(\boldsymbol{x}) z^k,$$

where $\alpha_1, \cdots, \alpha_n$ are fictitious indeterminates such that $x_i = (\alpha_1^i + \cdots + \alpha_n^i)/i$ and the polynomials $h_k(\boldsymbol{x})$ are the kth complete symmetric functions of $\alpha_1, \cdots, \alpha_n$ (see section **6.3.3** for the details of symmetric functions). Hence the solutions to (1.4) are the S-functions such that

$$s_\lambda(\boldsymbol{x}) = \det(h_{\lambda_j - i + j}(\boldsymbol{x})),$$

for any partition λ.

1.1.2 The BKP equation and the Schur Q-functions

In 1981 Date *et al* [32] discovered a new hierarchy which they called the KP hierarchy of B type (BKP for short). The BKP equations are the specialization of the KP equations. In partition notation the Hirota derivatives are restricted to odd part partitions only. Then

$$(D_{1^6} - 5D_{31^3} - 5D_{3^2} + 9D_{51})\tau \cdot \tau = 0, \tag{1.6}$$

is the BKP equation in Hirota form.

It was shown in [32] that the polynomial solutions of (1.6) are

$$\tau = L_\lambda(2\boldsymbol{x}; -1),$$

for any partition λ into distinct parts, where the symmetric functions $L_\lambda(2\boldsymbol{x}; -1)$ are directly related with the Schur Q-functions by a constant such that

$$L_\lambda(2\boldsymbol{x}; -1) = 2^{|\lambda|} Q_\lambda(\boldsymbol{x}; -1).$$

From now on we will ignore the constant factor $2^{|\lambda|}$ for simplicity and will treat $L_\lambda(2\boldsymbol{x}; -1)$ as a Schur Q-function and will denote it by Q_λ.

Recently Nimmo [19] has used the Pfaffian expansion of the Q-functions for the solution of the BKP equation and has developed a form analogous to the Wronskian representation of solutions to the KP equation. It has been shown that a Q-function Q_λ can be expanded in terms of its projections q_λ by using the differential operators

$$\prod(\partial^i; -1) = \prod_{i<j}(\partial^i - \partial^j)(\partial^i + \partial^j)^{-1},$$

similar to the raising operators [18],

$$\prod_{i<j}\frac{(1-\delta_{ij})}{(1+\delta_{ij})}.$$

Thus

$$Q_\lambda(\boldsymbol{x};-1)=\prod(\partial^i;-1)\prod_{i=1}^{l} q_{\lambda_i}(\boldsymbol{x};-1).$$

The polynomial solutions $L_\lambda(2\boldsymbol{x};-1)$ of the BKP equations can be generalized by considering a set of functions $\varphi_1(\boldsymbol{x}),\cdots,\varphi_n(\boldsymbol{x})$ depending on $\boldsymbol{x}=(x_1,x_2,x_3,\cdots)$ and satisfying (1.5). In terms of the Pfaffian expansion one can write the polynomial solutions as

$$\mathrm{Pf}(\varphi_1,\cdots,\varphi_n)=\prod_{i<j}(\partial^i-\partial^j)(\partial^i+\partial^j)^{-1}\prod_{k(odd)=1}^{n}\varphi_k(\boldsymbol{x})^k\,|_{x^k=x},$$

where 'Pf' stands for the Pfaffian of its argument. This result immediately gives

$$W_{BKP}(\varphi_1,\cdots,\varphi_N)=\prod_{i<j}(\partial^i+\partial^j)^{-1}W_{KP}(\varphi_1,\cdots,\varphi_N),$$

where the subscripts BKP and KP are used for the respective Wronskian determinants. The above equation can be compared with

$$S_\lambda(\boldsymbol{x};-1)=\prod_{i<j}(1+\delta_{ij})Q_\lambda(\boldsymbol{x};-1). \tag{1.7}$$

1.2 Quantum Hall effect and symmetric functions

1.2.1 The ordinary Hall effect

The ordinary Hall effect was discovered more than a century ago. It provides the simplest theory of conductivity in semiclassical form. It is based on the concept of mean free time τ_0 or mean free path l_0. In this picture, an electron of charge $-e$ and Fermi velocity v_F goes an average distance $l_0=v_F\tau_0$ before scattering to a new velocity whose average is zero. If the electron of mass m is in a weak electric field $\boldsymbol{E}$ it will on the average pick up an incremental velocity $\boldsymbol{\Delta v}=-e\boldsymbol{E}\tau_0/m$ before each scattering. Adding up the contribution of all the electrons leads to a current density $\boldsymbol{j}=\sigma_0\boldsymbol{E}$, with σ_0 is the conductivity and is given as

$$\sigma_0=\frac{ne^2\tau_0}{m},$$

where n is the electron number density.

Quantum effects enter through the band structure which changes the electron mass m_e to an effective mass m. At this point the Pauli principle is also applied which requires that only the electrons very near the Fermi surface are active.

In the presence of a magnetic field $\boldsymbol{B}$, the electron's path is curved due to the Lorentz force $-e\boldsymbol{v} \times \boldsymbol{B}/c$. Taking this into account one can write the current density as

$$\boldsymbol{j} = \sigma_0 \boldsymbol{E} - \sigma_0 \boldsymbol{j} \times \boldsymbol{B}/nec.$$

A specialization to the case of two dimensions, (xy) embedded in ordinary three-dimensional space, with $\boldsymbol{B}$ in the z-direction gives the resistivity tensor in the following form;

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_0 & B/nec \\ -B/nec & \rho_0 \end{bmatrix},$$

where $\rho_0 = 1/\sigma_0$. The conductivity tensor $\boldsymbol{\sigma}$ is the matrix inverse of $\boldsymbol{\rho}$ with components given by

$$\sigma_{xx} = \frac{\sigma_0}{1+\omega_c^2\tau_0^2}, \sigma_{xy} = -\frac{nec}{B} + \frac{1}{\omega_c\tau_0}\sigma_{xx},$$

where the cyclotron frequency $\omega_c = eB/mc$.

1.2.2 The quantum Hall effect

In 1980, Klaus von Klitzing discovered the quantum Hall effect and was awarded the 1985 Nobel prize in physics. It was suggested that under certain conditions in an effectively two-dimensional system of electrons subjected to a strong magnetic field, the conductivity tensor takes the form

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & -ie^2/h \\ ie^2/h & 0 \end{bmatrix},$$

where h is the Planck's constant and i is a small integer. This indicates that the current density $\boldsymbol{j}$ is directed precisely perpendicular to the electric field $\boldsymbol{E}$ as

$$j_i = \sum_j \sigma_{ij} E_j,$$

where the conductivity tensor has the quantized magnitude

$$\begin{aligned} |\sigma_{xy}| &= \frac{j}{E}, \\ &= \frac{ie^2}{h}. \end{aligned}$$

Thus the off-diagonal conductivity is given in terms of fundamental constants whereas the diagonal conductivity vanishes, so the system is dissipationless and thus is related to superconductivity and superfluidity.

In a series of papers [43, 44] it has been shown that all the current flows on the edges of the sample. It is because in the steady state the Lorentz and Coulomb forces would be exactly opposed and would only allow the electrons to enter and exit at diagonally opposite corners. Quite recently Michael Stone [28] has described how symmetric functions and Young diagrams provide a natural description of the excited states at the edge of the Hall droplet.

1.2.3 S-functions and many-body wavefunctions

Let us consider a two dimensional electrons gas (2DEG) moving in a perpendicular magnetic field $\boldsymbol{B}$. A normalized basis for the space of one-particle localized lowest Landau-level states is given by

$$\psi_n(x) = \frac{1}{\sqrt{\pi n!}} x^n e^{-\frac{B}{2}|x|^2}.$$

The simplest N-body state, $|0\rangle$, a homogeneous circular droplet of spin polarized electron liquid, is constructed out of the one-body state as a Slater determinant. Its wavefunction $\Psi_0(x) = \langle x|0\rangle$, factorises as the product of a Van der Monde determinant,

$$D(x) = \prod_{1\leq n<m\leq N} (x_n - x_m),$$

and a Gaussian

$$\Psi_0(x_1, x_2, \cdots, x_N) = \begin{vmatrix} x_1^{N-1} & x_1^{N-2} & \cdots & 1 \\ x_2^{N-1} & x_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_N^{N-1} & x_N^{N-2} & \cdots & 1 \end{vmatrix} e^{-\frac{B}{2}\sum_i |x_i|^2}.$$

All other N-body states in the same Landau level may be created by moving some electrons to orbits outside the droplet, and any wavefunction of this sort may be written as

$$\Psi_\lambda(x_1, x_2, \cdots, x_N) = \begin{vmatrix} x_1^{\lambda_1+N-1} & x_1^{\lambda_2+N-2} & \cdots & x_1^{\lambda_N} \\ x_2^{\lambda_1+N-1} & x_2^{\lambda_2+N-2} & \cdots & x_2^{\lambda_N} \\ \vdots & \vdots & \ddots & \vdots \\ x_N^{\lambda_1+N-1} & x_N^{\lambda_2+N-2} & \cdots & x_N^{\lambda_N} \end{vmatrix} e^{-\frac{B}{2}\sum_i |x_i|^2},$$

where λ is a partition.

The wavefunction $\Psi_\lambda(x) \equiv \langle x|\{\lambda\}\rangle$ still describes an N electron state but has $\lambda_1 + \lambda_2 + \cdots \lambda_N = |\lambda|$ extra powers of x. Each λ_i means that an electron has been moved from its position in the Fermi sea and raised from, say, the x^k state to the $x^{k+\lambda_i}$ state. If the droplet were confined in a potential well, the states near the edge would see a potential gradient and, since the single particle states are localized near a circle whose radius is proportional to $\sqrt{k}$, the new many-body state will have energy $E = c|\lambda|$ where c is proportional to the Hall drift velocity. The Hall droplet becomes a physical realisation of a chiral Dirac sea where all the states near the Fermi surface, identified with the physical edge of the droplet, move in same direction.

The ratio

$$s_\lambda(x) = \frac{\Psi_\lambda(x)}{\Psi_0(x)},$$

is an S-function. Sometimes an S-function s_λ is represented by a Young diagram of shape λ. For example,

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

represents s_{4221}. This pictorial representation has a very meaningful physical interpretation. For example, if we take the topmost electron and move it up, say, four steps, then the state can be represented by the Young diagram as

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \end{array}$$

but the same state could be thought of as being made by moving the bottom four empty states, or holes, down one place each. This leaves a gap filled by the particle, between these four holes and the rest. So from the hole viewpoint the state would be represented as

$$\begin{array}{l} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

i.e the conjugate diagram obtained by interchanging the rows and columns of the Young diagram. Hence the particle-hole interchange is seen as rows and column interchange.

An S-function can be regarded as an operator that acts on the wavefunction by multiplication and, for example, takes $|0\rangle$ to $|\{\lambda\}\rangle$ as shown below.

$$|\{\lambda\}\rangle = s_\lambda|0\rangle$$

The action of the operator s_λ on a general state $|\{\nu\}\rangle$ is to yield the sum of states obtained by combining the ν and λ diagrams via the Littlewood-Richardson rule. This combination is the outer product of S-functions. We can write

$$s_\lambda|\{\nu\}\rangle = \sum_\mu c^\mu_{\lambda\nu}|\{\mu\}\rangle,$$

where the coefficients $c^\mu_{\lambda\nu}$ are non-negative integers determined by the Littlewood-Richardson rule. As an example,

$$\begin{aligned}
s_{431}|\{521\}\rangle \; = \; & |\{952\}\rangle + |\{9511\}\rangle + |\{943\}\rangle + 2|\{9421\}\rangle + |\{94111\}\rangle + |\{9331\}\rangle \\
& + |\{9322\}\rangle + |\{93211\}\rangle + |\{862\}\rangle + |\{8611\}\rangle + 2|\{853\}\rangle \\
& + 4|\{8521\}\rangle + 2|\{85111\}\rangle + |\{844\}\rangle + 4|\{8431\}\rangle + 3|\{8422\}\rangle \\
& + 4|\{84211\}\rangle + |\{841111\}\rangle + 2|\{8332\}\rangle + 2|\{83311\}\rangle + 2|\{83221\}\rangle \\
& + |\{832111\}\rangle + |\{763\}\rangle + 2|\{7621\}\rangle + |\{76111\}\rangle + 2|\{754\}\rangle \\
& + 5|\{7531\}\rangle + 3|\{7522\}\rangle + 4|\{75211\}\rangle + |\{751111\}\rangle + 3|\{7441\}\rangle \\
& + 5|\{7432\}\rangle + 5|\{74311\}\rangle + 4|\{74221\}\rangle + 2|\{742111\}\rangle + |\{7333\}\rangle \\
& + 3|\{73321\}\rangle + |\{733111\}\rangle + |\{73222\}\rangle + |\{732211\}\rangle + |\{664\}\rangle \\
& + 2|\{6631\}\rangle + |\{6622\}\rangle + |\{66211\}\rangle + |\{655\}\rangle + 4|\{6541\}\rangle \\
& + 4|\{6532\}\rangle + 4|\{65311\}\rangle + 2|\{65221\}\rangle + |\{652111\}\rangle + 3|\{6442\}\rangle \\
& + 3|\{64411\}\rangle + 2|\{6433\}\rangle + 5|\{64321\}\rangle + 2|\{643111\}\rangle + |\{64222\}\rangle \\
& + |\{642211\}\rangle + |\{63331\}\rangle + |\{63322\}\rangle + |\{633211\}\rangle + |\{5551\}\rangle \\
& + 2|\{5542\}\rangle + 2|\{55411\}\rangle + |\{5533\}\rangle + 2|\{55321\}\rangle + |\{553111\}\rangle \\
& + |\{5443\}\rangle + 2|\{54421\}\rangle + |\{544111\}\rangle + |\{54331\}\rangle + |\{54322\}\rangle \\
& + |\{543211\}\rangle .
\end{aligned}$$

The S-functions form a basis of a Hilbert space. This Hilbert space turns out to be isomorphic to the quantum mechanical many-body space. Using the algebraic properties of the space of symmetric functions one can examine the action of the loop group of local gauge transformations on the Hall droplet. This group creates charged *soliton* like states which move along the droplet edge at the Hall drift velocity.

1.3 Vertex operators

Vertex operators were first introduced in *dual resonance models* in connection with describing particle interactions at a *vertex* and later, representations of affine Lie algebras by means of vertex operators have played an important role in string theory.

Vertex operators appear in string theory [36] as operators of the form

$$V(z) =: e^{i\beta\varphi(z)} :,$$

where $\varphi(x)$ is a bosonic field:

$$\varphi(z) = q_0 - p_0 z + i \sum_{n \neq 0} \frac{a_n}{n} e^{inz},$$

and the operators a_k obey

$$[a_n, a_m] = n\delta_{n+m,0}, \qquad [q_0, p_0] = i.$$

It is worth noting that a_k can be regarded as the power sum symmetric functions

$$p_k = \sum_i z_i^k,$$

or

$$n\frac{\partial}{\partial p_k},$$

depending on whether n is greater or less than zero.

Vertex operators appear with slight variations in different branches of mathematics and physics but basically they are a combination of a *creation operator* and an *annihilation operator.* Their order is important. For example in the description of quantum Hall effect [28], the vertex operators appear in a simplified form as

$$V(z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}p_n e^{-inz}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}e^{inz}\frac{\partial}{\partial p_n}\right), \tag{1.8}$$

and its conjugate

$$V^*(z) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}p_n e^{-inz}\right)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}e^{inz}\frac{\partial}{\partial p_n}\right). \tag{1.9}$$

The first part of (1.8) is regarded as a creation operator whereas the second part as an annihilation operator. It is also noted that these are nothing but the fermion fields. This fact can be illustrated by the action of $V(z)$ on the ground state $|0\rangle$. If we expand the creation operator part

$$\exp\sum_{n=1}^{\infty}\frac{1}{n}p_n e^{-inz} = 1 + h_1 e^{-iz} + h_2 e^{-2iz} + \cdots,$$

where h_n is a complete symmetric function and corresponds to the Young diagram with n nodes all in a single row in other words, to the state with a single fermion promoted n steps.

For the action of $V^*(z)$ we use the expansion

$$\exp-\sum_{n=1}^{\infty}\frac{1}{n}p_n e^{-inz} = 1 - a_1 e^{-iz} + a_2 e^{-2iz} - \cdots,$$

where a_n corresponds to the Young diagram with n nodes all in a single column. This diagram indicates that a single hole has been created n steps deep in the Fermi sea. The $(-1)^n$ in the above equation comes from commuting the relevant annihilation operator past n creation operators to reach its partner.

1.3.1 Twisted and untwisted vertex operators

Vertex operators can be defined with the help of infinite dimensional Heisenberg algebras [21, 25]. If we introduce a parameter, say, t in the Heisenberg algebra then the resulting vertex operators are regarded as *deformed* vertex operators.

Let $\mathcal{G}$ be the Heisenberg algebra generated by a and $\alpha_n, n \in \mathcal{Z}\setminus 0$, and satisfy the following relations;

$$[\alpha_m, \alpha_n] = \frac{m}{1-t^{|m|}}\delta_{m+n,0}a, \quad [\alpha_m, a] = 0,$$

then $\mathcal{G}$ has the canonical representation with $a = 1$ realized on the space $S(\mathcal{G}^-)$, the symmetric algebra generated by $\alpha_{-n}, n \in \mathcal{N}$.

Here we regard α_{-n} as multiplication operators on $S(\mathcal{G}^-)$ and α_n as annihilation operators $\frac{n}{1-t^n}\frac{\partial}{\partial \alpha_{-n}}$. The action of these operators can be illustrated as

$$\alpha_n \alpha_{-n}.1 = \frac{n}{1-t^n}, \qquad n \in \mathcal{N}.$$

Vertex operators on $S(\mathcal{G}^-)$ can be defined as follows:

$$\begin{aligned} U(x) &= \exp\left\{\sum_{n\geq 1}\frac{1-t^n}{n}\alpha_{-n}x^n\right\}\exp\left\{-\sum_{n\geq 1}\frac{1-t^n}{n}\alpha_n x^{-n}\right\}, \\ &= \sum_{n\in\mathbf{Z}} U_n x^{-n}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} U^*(x) &= \exp\left\{-\sum_{n\geq 1}\frac{1-t^n}{n}\alpha_{-n}x^n\right\}\exp\left\{\sum_{n\geq 1}\frac{1-t^n}{n}\alpha_n x^{-n}\right\}, \\ &= \sum_{n\in\mathbf{Z}} U_n^* x^{n}. \end{aligned} \tag{1.11}$$

A hermitian structure $\langle,\rangle$ in the space $S(\mathcal{G}^-)$ requires

$$\alpha_n^* = \alpha_{-n},$$

and defined as

$$\langle \alpha_{-n}, \alpha_{-n}\rangle = \frac{n}{1-t^n}, \qquad n \in \mathcal{N},$$

and more generally

$$\langle \alpha_{-\mu}, \alpha_{-\nu}\rangle = z_\mu(t)\delta_{\mu\nu},$$

where μ and ν are two partitions and $z_\mu(t)$ is defined as

$$z_\mu(t) = \prod i^{m_i} m_i! \prod_{j\geq 1}(1-t^{\mu_j})^{-1}, \quad \mu = (1^{m_1}2^{m_2}\ldots).$$

The specializations $t = 0, -1$ give untwisted and twisted vertex operators. The twisted vertex operators are restricted to odd integers only. This form of *twisted* vertex operators is different from the one used in the description of quantum affine Lie algebras in a sense that it is only a parameter *twisting.*

1.3.2 Vertex operators and symmetric functions

In section **1.3** we have noticed that the symmetric functions play a significant role in the description of vertex operators. In fact the space of vertex operators $\mathcal{V}$ and the ring of symmetric functions Λ_Q are isomorphic.

Let us introduce the following notation;

$$\exp\left(\sum_{n\geq 1}\frac{1-t^n}{n}\alpha_{-n}x^n\right) = \sum_{n\geq 0} R_n x^n,$$

where R_n's are considered as polynomials in α_n's. A general formula of R_n is given as

$$R_n = \sum_{|\lambda|=n}\frac{1}{z_\lambda(t)}\alpha_{-\lambda}, \qquad \alpha_{-\lambda} = \alpha_{-\lambda_1}, \alpha_{-\lambda_2}, \cdots, \alpha_{-\lambda_l},$$

where the sum runs over all the partitions λ of n.

The action of the components of the vertex operators $U(x)$ as defined in Eq. (1.9), can be shown as

$$U_{-n}.1 = \frac{1}{2\pi i}\int_c \exp\left(\sum_{m\geq 1}\frac{1-t^m}{m}\alpha_{-m}x^m\right) x^{-n}\frac{dx}{x},$$

where $i = \sqrt{-1}$ and the subscript c is for the contour integral.

It is easy to see that

$$U_{-n}.1 = R_n.$$

This can be generalized for any partition $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_l)$ such that

$$U_{-\lambda_1}U_{-\lambda_2}\cdots U_{-\lambda_l}.1 = \prod_{i<j}\frac{(1-\delta_{ij})}{(1-t\delta_{ij})}R_{\lambda_1}R_{\lambda_2}\cdots R_{\lambda_l}, \tag{1.12}$$

where δ_{ij} is the raising operator, $U_{-\lambda}.1 = U_{-\lambda_1}U_{-\lambda_2}\cdots U_{-\lambda_l}.1$ and $R_\lambda = R_{\lambda_1}R_{\lambda_2}\cdots R_{\lambda_l}$.

A characteristic mapping $\jmath : \mathcal{V} \to \Lambda_Q$ is defined as

$$\jmath(\alpha_{-\lambda}) = p_\lambda,$$

where λ is a partition and $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_l}$ is a power sum symmetric function defined in section **1.3**.

Under the map $\jmath : \mathcal{V} \to \Lambda_Q$, equation (1.12) reduces to

$$\jmath U_{-\lambda} = \prod_{i<j}\frac{(1-\delta_{ij})}{(1-t\delta_{ij})}q_{\lambda_1}q_{\lambda_2}\cdots q_{\lambda_l},$$

where

$$Q_\lambda(x;t) = \prod_{i<j}\frac{(1-\delta_{ij})}{(1-t\delta_{ij})}q_{\lambda_1}q_{\lambda_2}\cdots q_{\lambda_l},$$

are Hall-Littlewood symmetric functions and q_r and $q_\lambda = q_{\lambda_1}q_{\lambda_2}\cdots q_{\lambda_l}$ are defined as

$$\prod_i\left(\frac{1-tx_iy}{1-x_iy}\right) = \sum_{r=0}^{\infty} q_r(x;t)y^r,$$

and

$$q_\lambda(x;t) = \prod_i q_{\lambda_i}(x;t),$$

where y is a fictitious variable. If we set $t=-1$ then the above symmetric functions are nothing but the Schur Q-functions $Q_\lambda(-1)$ and for $t=0$ they reduce to S-functions s_λ .

1.4 The present work

The main objectives of this thesis are to explore some important properties of the symmetric functions, to develop the techniques for explicit calculations and to find some applications. The special emphasis is given to the Hall-Littlewood symmetric functions. In order to facilitate the explicit calculations, extensive computer programming was inevitable, which is done by extending SCHUR©, one of the most powerful software packages for group theory. A brief list of the topics covered in this thesis can be given as

- $O_n \downarrow S_n$ branching rules,
- Properties of Young raising operators,
- Properties of shifted tableaux,
- Properties of Q-functions ,
- Kronecker products of the spin irreps of S_n,
- The q-deformation of the symmetric functions and S_n,
- Applications to vertex operators.

Most of the present work is either published or submitted for publication in international journals [10, 17, 33, 34, 35].

1.4.1 $O_n \downarrow S_n$ branching rules

The orthogonal groups O_n and the symmetric groups S_n play an important role in nuclear and molecular models. Although the decomposition $O_n \downarrow S_n$ for ordinary irreps has been known for a long time the corresponding problem for spin irreps has received scant attention.

Following Morris [12] and Luan Dehuai and Wybourne [8] a simple method is developed for the embedding $O_n \downarrow S_n$ of ordinary and spin irreps in both n-dependent and in an n-independent reduced notation. The embedding of spin irreps is achieved by combining the basic spin irrep and ordinary irreps. The role of Young raising operators in the theory of symmetric functions has been exploited. A large number of Q-functions are generated as a result of the application of raising operators. This list contains null and nonstandard Q-functions along with the standard ones. The null Q-functions are screened and nonstandard ones are standarised by making use of the modification rules originally given by Morris. Computer implementation of these rules showed the need for a careful specification of these rules that would cater for all possibilities. As a result a complete and unambiguous set of modification rules for Q-functions is defined. All these techniques have been incorporated in the program SCHUR©. This work has been published in [10].

1.4.2 Properties of raising operators and Q-functions

There is a major disadvantage in using Young raising operators, that they generate a very large number of *dead* partitions which cannot be discarded during the process. Since these operators play a very important role in the manipulation of symmetric functions, it is worth simplifying the problem by exploring their properties.

A number of properties of Young's raising operator as applied to S-functions and Schur's Q-functions are noted. The order of evaluating the action of inverse raising operators is found to require careful specification and the maximum power of the operators δ_{ij} is determined. The operation of inverse raising operator on a partition λ is found to be the same as for its conjugate $\tilde{\lambda}$. A new definition of Shifted Lattice Property that can efficiently remove all the dead tableaux in the analogue of the Littlewood-Richardson rule for Q-functions is introduced. A simple combinatorial analogue of raising and inverse raising operators is given.

These newly discovered properties are used to write very simple computer algorithms for SCHUR©. This work has been submitted for publication to *Journal of Mathematical Physics* [33].

1.4.3 Shifted tableaux and Kronecker products of spin irreps of S_n

The importance of Young tableaux in the combinatorial description of S-functions is well known. Shifted Young tableaux play a similar role in the description of Schur's Q-functions and the projective representations of the symmetric group S_n. The shifted tableaux play a key role in the calculations of outer product and skew Q-functions. Because of their combinatorial nature, we again face the problem of a large number of *dead* tableaux. A study of their properties in connection with the shifted lattice property improved the algorithm for the calculation of outer product and skew Q-functions.

Although there is no direct method for the calculation of Q-function inner products, a simple algorithm is given by making use of the techniques developed earlier. Since the spin characters of the symmetric group S_n are directly related with the Q-functions , an algorithm for the Kronecker product of S_n spin irreps is given. These algorithms have been easily programmed in SCHUR© and the results are reported in the thesis . This work has been published in *Journal of Mathematical Physics* [17].

1.4.4 The q-deformation of the symmetric functions

The $q-$deformation of symmetric functions is introduced leading to $q-$analogues of many well-known relationships in the theory of symmetric functions. $q-$deformed scalar products are developed and used to define $q-$dependent symmetric functions. The symmetric functions commonly associated with the names Hall-Littlewood, Schur and Jack are all special cases of the $q-$deformation of Macdonald's new symmetric functions $P_\lambda(s,t)$. A $q-$analogue of the spin and ordinary characters of S_n is given and illustrated by the explicit calculation of examples of $q-$deformed characters. The methods used closely parallel those of quantum groups. This construction allows us to use the symmetric functions in the ares of physics involving q-deformation. The work on the q-deformation of symmetric functions is accepted for publication in *J. Phys. A:Gen. Math.*

[34].

1.4.5 q-Vertex Operators

A method similar to that used for the symmetric functions is developed for constructing the $q-$analogue of the vertex operators. A 1:1 correspondence between the space $\mathcal{V}$ of twisted $q-$vertex operators and the ring of $q-$deformed symmetric functions $\Lambda \otimes_{\mathcal{Z}} \mathcal{Q}(q,t)$ is established and a mapping from $\mathcal{V} \to \Lambda \otimes_{\mathcal{Z}} \mathcal{Q}(q,t)$ is defined. A number of relevant theorems are given.

The main idea is to define a q-analogue of the Heisenberg algebra with a parameter t, in order to construct the q-analogue of vertex operators. For $q \to 1$ these vertex operators are the ones used in dual resonance theory or in the description of affine Lie algebras. Using these q-vertex operators and the usual techniques of constructing twisted and untwisted vertex operators, we have constructed the twisted and untwisted q-vertex operators. The vertex operators so developed are in the most general form. The work on vertex operators is submitted to *J. Phys. A:Gen. Math.* [35].

1.4.6 Summary

Chapter 2 contains some notations, definitions and the properties of Young raising operators. In chapter 3 the properties of shifted tableaux are explored. Chapter 4 discusses some important properties of Q-functions and a simple algorithm for the calculation of Q-function inner product is given. The $O_n \downarrow S_n$ branching rules are discussed in chapter 5. Chapter 6 is devoted to the q-deformation of symmetric functions and the symmetric group. Finally in chapter 7 the techniques of chapter 6 are applied to vertex operators.

Chapter 2

Symmetric Functions

2.1 Introduction

A detailed account of the theory of symmetric functions can be found in [18, 20]. The purpose of this chapter is two fold, first to review some notations and terminology which will be used throughout the thesis and secondly to introduce, and make some comments on, the Schur Q-functions .
Sections 2 and 3 are devoted to a brief description of partitions and the permutation group S_n. Different types of symmetric functions are studied in section 4 and a very generalised form of symmetric function is given. The Schur Q-functions are discussed in section 5. Some very important properties of Young raising operators in connection with the S-functions and the Schur Q-functions are explored in section 6. Finally in section 7, a new set of modification rules for Q-functions is given.

2.2 Partitions

Partitions play a very important role in the theory of symmetric functions since all the symmetric functions are indexed by partitions. A partition is defined as

Definition 1 *Any (finite or infinite) sequence;*

$$\lambda = (\lambda_1 \lambda_2 ... \lambda_r ...),$$

of non-negative integers in non-increasing order;

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq ...,$$

and containing only a finite number of non-zero terms.

We do not distinguish between two such partitions which differ only by a string of zeroes at the end. For example, (31), (310), (3100...) are regarded as the same partition. The non-zero λ_i in the partition λ are called the parts of λ. The number of parts is the *length* of λ, denoted by $l(\lambda)$; and the sum of the parts is the *weight* of λ, denoted by $|\lambda|$:

$$|\lambda| = \sum_i \lambda_i.$$

If $|\lambda| = n$ then λ is called a partition of n and will be denoted as $\lambda \vdash n$. The set of all the partitions will be denoted by $\mathbf{P}$ and the set of all the partitions of n by $\mathbf{P}_n$. The set of all the partitions with distinct parts only is denoted by $\mathbf{DP}$ whereas $\mathbf{DP}_n$ is the set of all the distinct parts partitions of n. For example,

$$\mathbf{DP}_6 = \{(6), (51), (42), (321)\},$$

is the set of all the distinct parts partitions of 6.

Partitions are also represented by Young diagrams which will be discussed in detail in chapter 3. A Young diagram is defined as:

Definition 2 *The diagram of a partition λ is a set of points*

$$D_\lambda = \{(i,j) \in Z^2 : 1 \leq j \leq \lambda_i,\ 1 \leq i \leq l(\lambda)\},$$

in a plane with matrix-style coordinates.

For example, the diagram of the partition (421) is;

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

consisting of 4 points of nodes in the first row, 2 in the second row and 1 in the last row. The diagonal of the nodes in an Young diagram begining at the top left-hand corner is called the *main diagonal.*

The *conjugate* of a partition λ is the partition $\tilde{\lambda}$ whose diagram is obtained by reflection about the main diagonal. For example the conjugate of (421) is (3211) whose diagram is;

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array}$$

The *rank* of a partition is normally defined as *the number of nodes on the main diagonal* of the diagram but a more useful definition can be given as:

Definition 3 *The rank of a partition (ρ) is the maximum value of i for which $\rho_i \geq i$.*

For example the rank of (421) is 2.
In Frobenius notation, a partition λ is written as:

$$\lambda \equiv \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

and

$$\tilde{\lambda} \equiv \begin{pmatrix} b_1 & b_2 & \cdots & b_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

where r is the rank of the partition λ, a_i is the number of nodes to the right of the main diagonal in the i-th row and b_i is the number of nodes below the main diagonal in the i-th column. For example, the Young diagram of the partition (6432) is;

$$\begin{array}{llllll} \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \circ & \bullet & \bullet & & \\ \bullet & \bullet & \circ & & & \\ \bullet & \bullet & & & & \end{array}$$

and the rank is 3. It can be written in Frobenius notation as:

$$\lambda \equiv \begin{pmatrix} 5 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

It is important to note that the zeros are significant and must be retained.

2.3 The symmetric group S_n

The $n!$ permutations of the n objects $(a_1, a_2, ..., a_n)$ form a group known as the symmetric group and is denoted by S_n which will be discussed in detail in chapters 5 and 6. We can write a general form of a permutation as:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix},$$

where the n objects $(a_1, a_2, ..., a_n)$ denote the numbers (1,2,...,n) in some order, e.g.,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix},$$

i.e., $1 \to 4, 2 \to 1, 3 \to 3, 4 \to 2$. Permutations may be resolved into *cycles*, for example the above permutation may be regarded as the product of two closed cycles

$$(142)(3).$$

A cycle containing two symbols is called a *transposition.* Every cycle can be written as a product of transpositions. The number of transpositions is called the length of the cycle. The *cycle structure* of a permutation is the number of cycles of each length appearing in it when written as a product of independent cycles.

All the permutations in S_n with the same cycle structure belong to the same *class.* For example, in S_3 all the $3! = 6$ permutations are *classified* as:

$$\begin{array}{lll} (1)(2)(3); & & \\ (1)(23), & (12)(3), & (13)(2); \\ (231), & (312). & \end{array}$$

2.4 The ring of symmetric functions

Let $x_1, ..., x_n$ be independent indeterminates. The symmetric group S_n acts on the polynomial ring $\mathcal{Z}[x_1, ..., x_n]$ by permuting the x's, and we shall write [18],

$$\Lambda_n = \mathcal{Z}[x_1, ..., x_n]^{S_n},$$

for the subring of symmetric polynomials in $x_1, ..., x_n$.

Λ_n is a *graded* ring

$$\Lambda_n = \bigoplus_{r \geq 0} \Lambda_n^r,$$

where Λ_n^r is the additive group of symmetric polynomials of degree r in $x_1, ..., x_n$. Let

$$\Lambda^r = \varprojlim_n \Lambda_n^r,$$

for each $r \geq 0$, then

$$\Lambda = \bigoplus_{r \geq 0} \Lambda^r.$$

The graded ring Λ is the ring of symmetric functions. If $\mathcal{Q}$ is any commutative ring, we will write

$$\Lambda_{\mathcal{Q}} = \Lambda \otimes_{\mathcal{Z}} \mathcal{Q},$$

for the ring of symmetric functions with coefficients in $\mathcal{Q}$.

There are various $\mathcal{Z}$-bases of the ring Λ, and they all are indexed by partitions. Let $\lambda = (1^{m_1} 2^{m_2} \cdots)$ be a partition, where m_1 is number of parts equal to 1, m_2 is number of parts equal to 2, and so on. It defines a monomial $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$. The monomial symmetric function m_λ is the sum of all distinct monomials obtainable from x^λ by permutation of the x.

When $\lambda = (1^n)$ we have

$$m_{(1^n)} = e_n = \sum_{i_1 < ... < i_n} x_{i_1} \cdots x_{i_n},$$

the n-th elementary symmetric function.

For $\lambda = (n)$ we have

$$m_{(n)} = p_n = \sum_i x_i^n,$$

the n-th power sum symmetric function.

Let $l(\lambda)$ be the length of the partition λ and form the determinant

$$D_\lambda = \det \left(x_i^{\lambda_j + l - j} \right)_{1 \leq i,j \leq l(\lambda)},$$

and the Vandermonde determinant

$$D_\circ = \prod_{i<j} (x_i - x_j),$$

then the quotient

$$s_\lambda(x_1, ..., x_l) = D_\lambda / D_o,$$

is a homogeneous symmetric polynomial of degree $|\lambda|$ in $x_1, ..., x_l$ called S-functions. An S-function indexed by a partition λ is generally denoted by $\{\lambda\}$. S-functions indexed by disordered partitions are called *non-standard* S-functions and must be modified to produce either a signed standard partition or a null result by application of the following modification rules which derived directly from the determinantal definition of the S-functions [11].

i. In any S-function two consecutive parts may be interchanged provided that the preceding part is decreased by unity and the succeeding part increased by unity, the S-function thereby changed in sign, i.e.

$$\{\lambda_1, ..., \lambda_i, \lambda_{i+1}, ..., \lambda_l\} = -\{\lambda_1, ..., \lambda_{i+1} - 1, \lambda_i + 1, ..., \lambda_l\}.$$

ii. In any S-function if any part exceed by unity the preceding part the value of the S-function is zero, i.e. if

$$\begin{aligned} \lambda_{i+1} &= \lambda_i + 1, \\ \{\lambda\} &= 0. \end{aligned}$$

iii. The value of any S-function is zero if the last part is a negative number.

As an example,

$$\begin{aligned} \{154-12\} &= -\{15410\}, \\ &= \{4241\}, \\ &= -\{4331\}. \end{aligned}$$

A scalar product $\langle \, , \, \rangle$ is defined on Λ as follows:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda,$$

where

$$z_\lambda = \prod_i i^{m_i} m_i!.$$

Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ be two sequences of indeterminates over $Q(t)$, where t is another indeterminate and define [20],

$$\begin{aligned} P(x, y; t) &= \prod_{i,j} \frac{(1 - t x_i y_j)}{(1 - x_i y_j)}, \\ &= \sum_\lambda b_\lambda(t) P_\lambda(x; t) P_\lambda(y; t), \\ &= \sum_\lambda P_\lambda(x; t) Q_\lambda(y; t), \end{aligned}$$

as the generating function of *Hall-Littlewood symmetric functions* $P_\lambda(t)$. $Q_\lambda(y; t)$ is defined as

$$Q_\lambda(y; t) = b_\lambda(t) P_\lambda(y; t),$$

where

$$b_\lambda(t) = \prod_{i\geq 1} \phi_{m_i(\lambda)}(t), \qquad \phi_n(t) = \prod_{j\geq 1}^{n} \left(1 - t^j\right).$$

A scalar product $\langle\ ,\ \rangle_{(t)}$ over $\mathcal{Q}(t)$ is defined as follows:

$$\langle p_\lambda, p_\mu \rangle_{(t)} = \delta_{\lambda\mu} z_\lambda(t),$$

where

$$z_\lambda(t) = z_\lambda \prod_i (1 - t^{\lambda_i})^{-1}.$$

Let s be another independent indeterminate and define a scalar product $\langle\ ,\ \rangle_{(s,t)}$ over $\mathcal{Q}(s,t)$ as follows:

$$\langle p_\lambda, p_\mu \rangle_{(s,t)} = \delta_{\lambda\mu} z_\lambda(s,t), \tag{2.1}$$

where

$$z_\lambda(s,t) = z_\lambda \prod_i \frac{(1 - s^{\lambda_i})}{(1 - t^{\lambda_i})}. \tag{2.2}$$

The symmetric functions commonly associated with the names Hall-Littlewood, Schur and Jack are all special cases of this generalised symmetric function.

2.5 Schur's Q-functions

In 1911 Schur [1] introduced Q-functions in the development of the theory of the spin irreps of S_n (see chapter 5). A combinatorial definition of Schur's Q-functions will be discussed in chapter 3. Schur's Q-functions are the specialisation $s = 0$ and $t = -1$ of the above mentioned generalised symmetric functions [13, 17]. Schur's Q-functions will be denoted by P_λ for a partition λ.
Let us introduce another symmetric function Q_λ related to P_λ as follows:

$$Q_\lambda = 2^{l(\lambda)} P_\lambda, \tag{2.3}$$

so that

$$\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda\mu},$$

i.e., P_λ, Q_λ are dual bases of Λ_Q for the scalar product $\langle\ ,\rangle$.

A relationship between S-functions and the ordinary characters of the symmetric group S_n is given as

$$s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^{[\lambda]} p_\rho,$$

where $\{\lambda\}$ is an S-function corresponding to the partition λ and $\chi_\rho^{[\lambda]}$ is the character of S_n corresponding to the irreps $[\lambda]$ and the class ρ.

The connection between Schur Q-functions and the spin characters of the symmetric group S_n is made explicit in a similar fashion by Schur's relation [1],

$$Q_\lambda = 2^{(l(\lambda)+l(\pi)+\epsilon)/2} \sum_\pi z_\pi^{-1} \zeta_\pi^{[\Delta;\lambda]} p_\pi, \tag{2.4}$$

where $\zeta_\pi^{[\Delta;\lambda]}$ is a simple spin character of the class $\pi = 1^{\alpha_1} 3^{\alpha_3} \ldots$ involving odd cycles only and $\epsilon = 0$ or 1 according as $(n - l(\lambda))$ is even or odd.

2.6 Young Raising Operators

The operation of Young raising operator δ_{ij} on a partition $\lambda \equiv (\lambda_1 \cdots \lambda_i \cdot \lambda_j \cdots \lambda_n)$ increases λ_i and decreases λ_j by 1 provided $i < j$. Young raising operators play a very important role in the operations of generalised S-functions $S_\lambda(-1)$ and Schur's Q-functions Q_λ as shown in Eq. (1.7). From now on the generalised S-function, S_λ will be represented by $\{\lambda\}_g$, defined as

$$\{\lambda\}_g = \prod_{i<j} (1 - \delta_{ij})\, q_\lambda.$$

Here we will discuss their properties in this context which have not been discussed before.

One can expand a generalised S-function in terms of Q-functions using the raising operator as follows:

$$\{\lambda_1\lambda_2 \cdots \lambda_n\}_g = \prod_{1\le i<j\le n} (1 + \delta_{ij})Q_\lambda, \tag{2.5}$$

and a Q-function in terms of generalised S-functions using the inverse raising operator as follows:

$$Q_\lambda = \prod_{1\le i<j\le n} [1 + \sum_t (-1)^t \delta_{ij}^t]\{\lambda\}_g. \tag{2.6}$$

2.6.1 Order of the Inverse Raising Operators

It is generally assumed that the order of the operators is not important. Actual applications of (2.5) and (2.6) show that the order of raising operators is not important indeed but that of inverse raising operators used in (2.6) is. We observe that the operators in (2.6) should be set in such a way that from right to left the values of j are in non-ascending order and the values of i are in descending order for a given value of j. As an example, for a three part partition λ, the equation (2.6) becomes

$$Q_\lambda = [1 + \sum_t (-1)^t \delta_{12}^t][1 + \sum_t (-1)^t \delta_{13}^t][1 + \sum_t (-1)^t \delta_{23}^t]\{\lambda\}_g. \tag{2.7}$$

2.6.2 Maximum Power of the Operators δ_{ij}

Regarding the maximum value of t we observe that the list on the right side of the equation (2.6) contains S-functions, so the maximum value of t for each operator δ_{ij}^t is that value for which $\lambda_j - t = j - l(\lambda)$. Hence it is not only dependent on the value of λ_j as given by Thomas [16] but also on the position of λ_j relative to the length of λ. As an example, if we apply $[1 + \sum_t (-1)^t \delta_{12}^t]$ to $\{321\}_g$ then the maximum value of t will be 3 whereas according to Thomas it should not be greater than 2. Thus

$$\begin{aligned}(1 - \delta_{12} + \delta_{12}^2 - \delta_{12}^3)\{321\}_g &= \{321\}_g - \{411\}_g + \{501\}_g - \{6-11\}_g, \\ &= \{321\}_g - \{411\}_g + \{6\}_g,\end{aligned} \tag{2.8}$$

where the second line of (2.8) is achieved by using the modification rules for S-functions.

2.6.3 Raising Operator and Conjugate Partitions

Another important property of the raising operator used in equation (2.5) which simplifies its computation is that it gives the same set of Q-functions if applied on $Q_{\tilde{\lambda}}$ where $\tilde{\lambda}$ is the conjugate partition of λ. As an example the Q-function content of $\{5\}_g$ is same as that of $\{1^5\}_g$. It is clear from the S_n irreps analogue that the characters of positive classes of conjugate representations are same. Hence the expansion of conjugate representations in terms of spin irreps will also be same.

2.6.4 Reduced Notation

Murnaghan [6, 7] introduced a reduced notation for the labelling of irreps of S_n for an n-independent resolution of the Kronecker products. In the reduced notation, the irrep of S_n usually labelled by the symbol $[\lambda] \equiv [n-m, \mu_1, \mu_2, ...]$, with (μ) being a partition of m, is labelled by the symbol $\langle\mu\rangle_g \equiv \langle\mu_1, \mu_2, ...\rangle$.

In order to facilitate the reduced notation, Luan and Wybourne [8] introduced a special Young raising operator δ_{0j}, which in the reduced notation has the effect of decreasing μ_i by one unit. Hence the equations (2.5) and (2.6) can be rewritten in reduced notation as follows:

$$\langle\mu\rangle_g = \prod_{0\leq i<j\leq n} (1+\delta_{ij})Q_\mu, \tag{2.9}$$

and

$$Q_\mu = \prod_{0\leq i<j\leq n} [1+\sum_t(-1)^t\delta_{ij}^t]\langle\mu\rangle_g. \tag{2.10}$$

The properties discussed in sections **2.6.2** and **2.6.3** are not applicable to the operators in reduced notation.

2.7 Modification Rules

The list of Q-functions produced at the right hand side of the equation (2.5) usually contain non-standard partitions. These non-standard partitions can be converted into standard ones by using modification rules initially given by Morris [9] and later on completed and modified by Salam and Wybourne [10]. These rules are based on the following recurrence relations obeyed by the Q-functions (see section **1.3**), originally given by Schur.

$$\begin{aligned} Q_{\lambda_1\lambda_2\cdots\lambda_l} &= Q_{\lambda_1\lambda_2}Q_{\lambda_3\lambda_4\cdots\lambda_l} - Q_{\lambda_1\lambda_3}Q_{\lambda_2\lambda_4\cdots\lambda_l} \\ &+\cdots+ Q_{\lambda_1\lambda_l}Q_{\lambda_2\lambda_3\cdots\lambda_{l-1}} \qquad (l \text{ even}), \end{aligned}$$

and

$$\begin{aligned} Q_{\lambda_1\lambda_2\cdots\lambda_l} &= q_{\lambda_1}Q_{\lambda_2\lambda_3\cdots\lambda_l} - q_{\lambda_2}Q_{\lambda_1\lambda_3\cdots\lambda_l} \\ &+\cdots+ q_{\lambda_l}Q_{\lambda_2\lambda_3\cdots\lambda_{l-1}} \qquad (l \text{ odd}), \end{aligned}$$

and

$$Q_{\lambda_1\lambda_2} = q_{\lambda_1}q_{\lambda_2} - 2q_{\lambda_1+1}q_{\lambda_2-1} + \cdots + (-1)^{\lambda_2}q_{\lambda_1+\lambda_2}.$$

Any list of Q-functions may be converted into a list of standard Q-functions by sequential application of the following four rules to the list.

i. The parts of the Q-function are first ordered so that the absolute magnitude of the parts are in descending order when read from left to right. This is achieved by repeated use of

$$Q_{(\ldots\lambda_i,\lambda_{i+1},\ldots)} = -Q_{(\ldots\lambda_{i+1},\lambda_i,\ldots)},$$

whenever $\mid \lambda_{i+1} \mid > \mid \lambda_i \mid$, remembering that $Q_{(\mu,0)} = Q_{(\mu)}$.

ii. Q-functions with consecutive repeated parts are null.

iii. Q-functions where a negative part $-\lambda_p$ precedes λ_p is modified by use of the identity

$$Q_{(\lambda_1\ldots-\lambda_p,\lambda_p\ldots\lambda_k)} = (-1)^{\lambda_p} 2Q_{(\lambda_1\ldots\lambda_k)}.$$

iv. Any remaining Q-function containing a negative part is null.

As an example, (i) leads to $Q_{(304-212-11)} \to Q_{(43-221-11)}$. Application of (iii) then leads to $Q_{(43-221-11)} \to -4Q_{(431)}$ and hence $Q_{(304-212-11)} \equiv -4Q_{(431)}$.

We note in the above example that it is possible to have repeated parts provided they are separated by a negative part of the same magnitude.

Chapter 3

Young Tableaux

3.1 Introduction

The importance of Young tableaux in the combinatorial description of S-functions is well known. The shifted Young tableaux play a similar role in the description of the Schur Q-functions and the projective representations of the symmetric group S_n (the irreps of S_n will be studied in detail in chapter 5). The theory of shifted tableaux has been developed by Sagan [4], Worley [5] and Stembridge [13]. The main purpose of this chapter is to explore some very important properties of skew shifted tableaux.

A brief review of the Young tableaux and their natural connection with the S-functions is given in section 2. The shifted Young tableaux are introduced in section 3 and a more useful definition of the shifted lattice property for computational purpose is given. The Schur Q-functions are described in terms of shifted tableaux in section 4. Section 5 is devoted to some new properties of skew shifted tableaux. These properties lead us to an easy computation of skew Q-function and Q-function outer products.

3.2 Young Tableaux and S-functions

Young tableaux associated with a Young diagram of shape λ are known as *ordinary tableaux* and defined as

Definition 4 *An ordinary tableau $T(\lambda)$ of shape λ is an assignment $T : D_\lambda \to P$ of letters from the ordered alphabet $P = \{1 < 2 < 3 < ...\}$ satisfying*

- $T(i,j) < T(i+1,j)$,
- $T(i,j) \leq T(i,j+1)$.

The first condition ensures the increasing columns whereas the second makes nondecreasing rows. For example,

$$\begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & & \\ 3 & 4 & 4 & & & \\ 5 & 6 & & & & \end{array}$$

is an ordinary tableau of shape (6432). Let ρ_k denote the number of points $(i,j) \in D_\lambda$ such that $T(i,j) = k$ then the tableaux $T(\lambda)$ is said to have content $\rho = (\rho_1, \rho_2, \rho_3, \cdots)$, and is written as $x^T = x_1^{\rho_1} x_2^{\rho_2} x_3^{\rho_3} \cdots$.

The combinatorial theory of tableaux is closely related with the theory of symmetric functions and is made evident by the following definition of the S-functions.

$$s_\lambda(x) = \sum_T x^T, \tag{3.1}$$

where the sum is over all the ordinary tableaux of shape λ. For example,

$$s_{21}(x) = \sum_{i \neq j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k.$$

3.2.1 Skew Diagram

Consider two partitions λ and μ such that $D_\lambda \supseteq D_\mu$ which implies

$$|\lambda| \geq |\mu|, \quad l(\lambda) \geq l(\mu), \quad \lambda_i \geq \mu_i \quad \forall i.$$

If the diagram D_μ is superimposed on that of D_λ then the points not covered by D_μ will form a *skew diagram* and will be designated as $D_{\lambda/\mu}$. In other words,

Definition 5 *A skew diagram is a collection of points of the form $D_{\lambda/\mu} := D_\lambda - D_\mu$, where D_μ and D_λ are any pair of diagrams such that $D_\lambda \supseteq D_\mu$.*

As an example,

$$\begin{array}{cccccc} & & & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & & \\ & \bullet & \bullet & & & \\ & \bullet & & & & \end{array}$$

is a skew diagram $D_{6432/3211}$. It is important to note that the same conditions apply to the skew tableaux as to ordinary tableaux. For example,

$$\begin{array}{cccccc} & & & 1 & 2 & 4 \\ & & 2 & 3 & & \\ & 4 & 4 & & & \\ & 5 & & & & \end{array}$$

is the skew tableau $T(6432/3211)$.

3.2.2 Lattice Permutation

Let T be a tableau then a *word* $w(T)$ is defined as the sequence of positive integers read from right to left in successive rows, starting with the top row. For example, if $T(643/221)$ is a skew tableau

$$\begin{array}{cccccc} & & 1 & 1 & 1 & 2 \\ & & 2 & 3 & & \\ 1 & 4 & & & & \end{array}$$

then $w(T) = 21113241$ is the word.

Definition 6 *A word $w = a_1 a_2 \cdots a_N$ in the symbols $1, 2, 3, \cdots, n$ is said to be a lattice permutation if for $1 \le r \le N$ and $1 \le i \le n-1$, the number of occurrences of the symbol i in $a_1 a_2 \cdots a_r$ is not less than the number of occurrences of $i+1$.*

Let k_r be the number of integers in the word $w(T)$ lying to the left of a_r which are equal to a_r, including a_r itself, minus the number of integers lying to the left of a_r which are equal to $a_r - 1$, then according to the above definition, $w(T)$ will be a lattice permutation if $k_r \le 0 \;\; \forall r$ such that $a_r > 1$. For example the above word $w(T) = 21113241$

r	1	2	3	4	5	6	7	8
a_r	2	1	1	1	3	2	4	1
k_r	1	0	2	3	0	−1	0	4

is not a lattice permutation whereas the word $w(T) = 121121323$

r	1	2	3	4	5	6	7	8	9
a_r	1	2	1	1	2	1	3	2	3
k_r	1	0	2	3	−1	4	−1	−1	−1

is a lattice permutation.

3.2.3 The Littlewood-Richardson Rule

The product of two S-functions s_μ and s_ν indexed by the partitions μ and ν respectively, is an integral combination of S-functions:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda, \tag{3.2}$$

where the coefficients $c^\lambda_{\mu\nu}$ are non-negative integers and $|\lambda| = |\mu| + |\nu|$ such that $\lambda \supseteq \mu, \nu$.

The coefficients $c^\lambda_{\mu\nu}$ can be calculated by making use of the character theory of the symmetric group S_n [20]. The Littlewood-Richardson rule gives a simple combinatorial method for the calculation of $c^\lambda_{\mu\nu}$ defined as:

Definition 7 *The coefficients $c^\lambda_{\mu\nu}$ is the number of tableaux T of shape $D_{\lambda/\mu}$ and content ν such that $w(T)$ is a lattice permutation.*

3.3 Shifted Young Tableaux

A shifted diagram is a diagonally adjusted Young diagram with the restriction that the $(i+1)$-th row does not exceed the i-th row. This condition ensures that partitions are restricted to those involving distinct parts only. Let $\mathbf{P}'$ denote the ordered alphabet $\{1' < 1 < 2' < 2 \cdots\}$. The letters $1', 2' \cdots$ are said to be *marked* and we denote an *unmarked* version of any $a \in \mathbf{P}'$ by $|a|$.

Definition 8 *For each $\lambda \in \mathbf{DP}$ there is an associated shifted diagram defined as*

$$\mathbf{D'}_\lambda = \{(i,j) \in Z^2 : i \leq j \leq \lambda_i + i - 1,\ 1 \leq i \leq \ell(\lambda)\}.$$

For example,

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & \\ & & & \bullet & \bullet & \end{array}$$

is the shifted diagram of shape (6432).

Definition 9 *A shifted tableau T of shape λ is an assignment $T : D'_\lambda \rightarrow P'$ satisfying the following conditions:*

1. $T(i,j) \leq T(i+1,j)$, $T(i,j) \leq T(i,j+1)$,
2. *Each column has at most one k ($k = 1, 2, \cdots$),*
3. *Each row has at most one k' ($k' = 1', 2' \ldots$).*

As an example,

$$\begin{array}{cccccc} 1' & 1 & 1 & 2' & 2 & 6' \\ & 2 & 2 & 2 & 5' & 7 \\ & & 3' & 5' & 5 & \\ & & & 5' & & \end{array}$$

is a shifted tableau of shape (6531).

3.4 Schur's Q-functions

Similar to the S-functions, the Schur Q-functions can be defined in terms of shifted tableaux. Let $Q_\lambda = Q_\lambda(x)$ be a generating function in the variables $x_1, x_2 \cdots$ for each $\lambda \in \mathbf{DP}$ such as

$$Q_\lambda = \sum_{T: D'_\lambda \rightarrow P'} x^T, \tag{3.3}$$

where summation is over all the shifted tableaux $T(\lambda)$. For example,

$$\begin{aligned} Q_1(x) &= 2\sum_i x_i, \\ Q_2(x) &= 2\sum_i x_i^2 + 4\sum_{i<j} x_i x_j, \\ Q_{21}(x) &= 4\sum_{i \neq j} x_i x_j^2 + 8 \sum_{i<j<k} x_i x_j x_k. \end{aligned}$$

3.5 Shifted Lattice Property

A shifted analogue of the lattice permutation is given by Stembridge [13]. Given a shifted tableaux T a word $w(T) = w_1 w_2 \cdots w_n$ is a sequence obtained by reading the rows of T from left to right (rather than right to left), starting with the last row (rather than the top row). Let $w^r = w_n w_{n-1} \cdots w_1$ denote the reverse of w and let $\hat{w} = \hat{w}_1 \cdots \hat{w}_n$ denote the word obtained by inverting the marks of w, i.e. $\hat{2} = 2'$ and $\hat{2'} = 2$. Let $n_i(w, j)$ denote the number of occurrences of the letter i among $w_1 \cdots w_j$ and $n_i(w, 0) = 0$. An extended word of T is the sequence defined by $e(T) = w^r \hat{w}$. The tableau T is said to satisfy the *shifted lattice property* if the extended word $e = e_1 \cdots e_{2n}$ satisfies the following conditions for all $i \geq 1$ and $0 \leq j < 2n$.

$$n_i(e, j) = n_{i-1}(e, j) \text{ implies } \begin{cases} e_{j+1} \neq i, i' & 0 \leq j < n \\ e_{j+1} \neq i, (i-1)' & n \leq j < 2n \end{cases} \tag{3.4}$$

We can modify Stembridge's definition of the shifted lattice property as follows.

Definition 10 *The tableaux T is said to satisfy the shifted lattice property if the reversed word $w^r = w_n w_{n-1} \cdots w_1$ satisfies the following conditions for all $i \geq 1$, $0 < j \leq n$ and $0 \leq k < n$.*

i $n_i(w^r, k) = n_{i-1}(w^r, k)$ *implies* $w^r_{k+1} \neq i, i'$,

ii $n_{(i-1)'}(w^r, j) - n_{i'}(w^r, j) = \nu_{(i-1)} - \nu_i$ *implies* $w^r_{j-1} \neq i', (i-1)$,
where ν is the content of the tableaux T.

The second condition in (3.4) arises when

$$n_i(e, n) + n_{i'}(e, n) - n_{i'}(e, 2n - j) = n_{(i-1)}(e, n) + n_{(i-1)'}(e, n) - n_{(i-1)'}(e, 2n - j),$$

whereas we know that $n_{|i|}(e, n) = \nu_i$ and $\nu_{i-1} \geq \nu_i$, hence

$$n_{(i-1)'}(e, 2n - j) - n_{i'}(e, 2n - j) = \nu_{i-1} - \nu_i.$$

Thus to satisfy the second condition of (3.4) $e_{2n-j-1} \neq i', (i - 1)$.

There are two advantages of this definition. First we do not need to form extended word as required by Stembridge. Secondly during the formation of the tableaux from top right corner which is the reverse direction of the word, the dead tableaux can easily be discarded as soon as they arise.

3.6 Properties of Shifted Skew Tableaux

The shifted skew tableaux play a very important role in the theory of Q-functions. It is worth exploring the properties of shifted skew tableaux in order to simplify the computational problems. Some new theorems and their proofs will be given in this section and will be used in chapter 4.

Definition 11 *A shifted skew tableau T of shape λ/μ is an assignment $T : D'_{\lambda/\mu} \to P'$ satisfying the usual shifted rules.*

Using this definition one can define a skew Q-function in terms of shifted skew tableaux as

$$Q_{\lambda/\mu} = \sum_{T:D'_{\lambda/\mu} \to P'} x^T, \tag{3.5}$$

where summation is over all the shifted skew tableaux $T(\lambda/\mu)$. It can also be shown that [20],

$$Q_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu} Q_{\nu} \tag{3.6}$$

such that $\mu, \nu, \lambda \in \mathbf{DP}$, $|\lambda| = |\mu| + |\nu|$, $\lambda \supseteq \mu, \nu$, and the coefficients $f^{\lambda}_{\mu\nu}$ are non-negative whole numbers. The same coefficients appear in the outer product of two Q-functions Q_μ and Q_ν such as

$$Q_\mu \cdot Q_\nu = \sum_{\lambda} 2^{[\ell(\mu)+\ell(\nu)-\ell(\lambda)]} f^{\lambda}_{\mu\nu} Q_{\lambda}. \tag{3.7}$$

The coefficients $f^{\lambda}_{\mu\nu}$ can be calculated using the *shifted analogue of the Littlewood-Richardson rule* given by Stembridge [13]:

Definition 12 *The coefficient $f^{\lambda}_{\mu\nu}$ is defined as the number of shifted tableaux T of shape λ/μ and content ν such that*

1. *T satisfies the shifted lattice property,*
2. *The leftmost i of $w(T)$ is unmarked $(1 \le i \le \ell(\nu))$.*

As an example, if $\lambda \equiv 6421$, $\mu \equiv 431$ and $\nu \equiv 32$, then we obtain the following three shifted tableau of shape $Q_{6421/431}$ and content 32 which satisfy the above conditions.

$$\begin{array}{ccc} & 1 & 1 \\ & 2 & \\ 1 & & \\ 2 & & \end{array} \qquad \begin{array}{ccc} & 1 & 1 \\ & 2' & \\ 1 & & \\ 2 & & \end{array} \qquad \begin{array}{ccc} & 1' & 1 \\ & 2' & \\ 1 & & \\ 2 & & \end{array}$$

Hence $f^{6421}_{431,32} = 3$. A partition $\nu = (\nu_1\nu_2 \cdots \nu_i)$ is *lower* than $\mu = (\mu_1\mu_2 \cdots \mu_j)$ if for all $1 \le k \le j, \nu_k \le \mu_k$ and $|\mu| = |\nu|$.

Theorem 1 *In a skew shifted tableau of shape λ/μ and content ν satisfying the shifted lattice property, no $|i|$ can be placed in the j-th row such that $|i| > j$.*

Proof

In the first part we prove that an entry $x > 1$ placed in the first row violates the shifted lattice property. In order to satisfy the conditions of shifted tableaux the largest entry $x > 1$ in the first row must be placed in the right most position. This entry will ultimately appear at the first position of the extended word $e = e_1 \cdots e_{2n}$. Noting (2) requires

$$n_x(e,0) = n_{x-1}(e,0) = 0,$$

but $e_1 = x$ which violates the shifted lattice property. Similarly if an entry $y > 2$ is placed in the second row we again obtain a violation of the shifted lattice property leading readily to the same conclusion for every row and hence theorem 1.

Use of theorem 1 makes it possible to eliminate most of the *dead tableaux* which do not satisfy lattice property.

Theorem 2 *In a skew shifted tableau of shape λ/μ and content ν for all $1 \leq i \leq \ell(\lambda)$,*

$$\sum_i \lambda_i \leq \sum_i (\mu_i + \nu_i). \tag{3.8}$$

Proof

If $\sum_i(\lambda_i - \mu_i)$ is greater than $\sum_i \nu_i$ for any value of i then we have to make $\sum_i(\lambda_i - \mu_i) - \sum_i \nu_i$ entries greater than i in the first i rows, which violates theorem 1.

Corollary 1 *The largest partition λ appearing in the expansion of the outer product of two Q-functions Q_μ and Q_ν is given by*

$$\lambda = \mu + \nu,$$

such that

$$\lambda_i = \mu_i + \nu_i, \qquad \forall\ i. \tag{3.9}$$

Proof

It is easily concluded from (3.4) that the maximum value of λ_i is obtained when only equality holds for all i.

Corollary 2 *In the outer product of two Q-functions Q_μ and Q_ν the coefficient $f^{\lambda}_{\mu\nu}$ for the highest partition $\lambda = \mu + \nu$ is equal to 1.*

Proof

It is obvious that in the shifted skew tableaux λ/ν the i-th row is equal to ν_i. From theorem 1 all the i's will go to the i-th row. As we are not allowed to mark the rightmost i in the reversed word (see sec. **3.2**), we get only one standard word.

Corollary 1 gives immediately the highest partition in the outer product of two Q-functions. We now use the relationship between skew shifted tableaux and the shifted lattice property to establish the lowest live partition in a Q-function outer product.

Theorem 3 *Let μ and ν be self-conjugate partitions, each with distinct parts.*

$$Q_\mu \cdot Q_\nu = 2^{min[\ell(\mu),\ell(\nu)]} Q_\lambda, \tag{3.10}$$

where $\lambda = \mu + \nu$ and $min[\ell(\mu), \ell(\nu)]$ is the minimum of the length of the partitions (μ) and (ν).

Proof

Let $\nu = (\nu_1\nu_2\cdots\nu_{n-1}\nu_n)$ be the content of the skew shifted tableaux of shape λ/μ. It is clear that $\nu_n = 1$ and $\nu_{n-1} = 2$. Let x denote ν_{n-1} entries and y denote ν_n. There are only six possible types of extended words consistent with condition 2 of Definition 12.
(i) $\cdots x' \cdots y \cdots x \cdots x' \cdots y' \cdots x \cdots$
(ii) $\cdots x \cdots y \cdots x \cdots x' \cdots y' \cdots x' \cdots$
(iii) $\cdots y \cdots x' \cdots x \cdots x' \cdots x \cdots y' \cdots$
(iv) $\cdots y \cdots x \cdots x \cdots x' \cdots x' \cdots y' \cdots$
(v) $\cdots x' \cdots x \cdots y \cdots y' \cdots x' \cdots x \cdots$
(vi) $\cdots x \cdots x \cdots y \cdots y' \cdots x' \cdots x' \cdots$
Let y and y' always appear in each of these extended words in the jth and kth positions, respectively. Then (i), (iii) and (iv) do not satisfy the shifted lattice property since

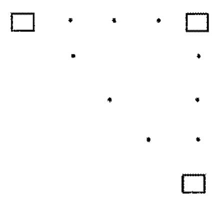
$$n_x(e, j-1) = n_y(e, j-1) = 0 \text{ but } e_j = y \text{ when } y = x+1.$$

(v) does not satisfy the shifted lattice property because

$$n_x(e, k-1) = n_y(e, k-1) = 1 \text{ but } e_k = y'.$$

The only surviving extended words are (ii) and (vi) both of which do not contain marked letters.

Successive application of these arguments for other parts show that in such a situation we cannot make any marked entry. As the partition μ is self-conjugate it forms a right adjusted shifted diagram as shown below.



Hence the skew diagram of shape λ/μ is a left adjusted diagram with ν_i number of boxes in i-th row for all i.

It should now be clear that if we are not permitted to make marked entries then all the 1's must be placed in the first row, all the 2's in the second row, etc. Thus $\lambda = \mu + \nu$. Since none of the entries can be marked or interchanged we conclude $f^{\lambda}_{\mu\nu} = 1$.

We have for example,

$$Q_{4321} \cdot Q_{4321} = 16Q_{8642} \qquad Q_{4321} \cdot Q_{54321} = 16Q_{97531}.$$

We note that for the S-function outer product $\{4321\} \cdot \{4321\}$ involves 206 distinct Young tableaux and a multiplicity sum of 930.

Extending the above arguments for any pair μ and ν we establish the following algorithm to yield the lowest partition that arises in $Q_\mu \cdot Q_\nu$. In order to work out the lowest partition ρ appearing in the outer product of two Q-functions Q_μ and Q_ν, we combine

μ and ν in non-ascending order i.e. $\mu \cup \nu \equiv \rho \equiv \rho_1 \cdots \rho_n$. If ρ is not a partition into distinct parts then the following algorithm can be used to achieve the desired partition.

Algorithm 1

1. Find the smallest i for which we get a sequence such that $\rho_i = \rho_{i+1}$ or $\rho_i = \rho_{i+1}+1$.
2. Find the number k of successive pairs of equal parts in this sequence; $\rho_j = \rho_{j+1} = \rho_{j+2}$ will be treated as one *pair*.
3. Add k to ρ_i, subtract the same from ρ_{i+1}

$$\text{i.e. } \rho \equiv \rho_1 \cdots \rho_i + k, \rho_i - k, \cdots \rho_n,$$

   and rearrange.
4. Repeat steps 1 to 3 until a partition into distinct parts is obtained.

As an example of $\mu = 541$ and $\nu = 542$, the lowest partition can be obtained by the use of the above algorithm.

$$\begin{aligned} \rho \equiv 554421 &\equiv 744321 && \text{using (1) to (3)} \\ &\equiv 753321 && \text{using (1) to (3)} \\ &\equiv 754221 && \text{using (1) to (3)} \\ &\equiv 754311 && \text{using (1) to (3)} \\ &\equiv 75432 && \text{using (1) to (3)} \end{aligned}$$

Corollary 1 gives the highest partition, which in this case is 10 83. It follows that the only live partitions are those partitions γ into distinct parts such that $10\,83 \geq \gamma \geq 75432$. Hence for any pair of distinct part partitions μ and ν one can form a complete set of live partitions in between the highest and the lowest partitions using the above algorithm. In the case of μ and ν being self conjugate partitions, each into distinct parts, the highest and the lowest live partitions coincide as expected.

Chapter 4

Properties and Products of Q-functions

4.1 Introduction

In the previous chapter we have noticed that a Q-function outer product can be evaluated by using the theory of shifted tableaux. This can also be done by expanding the two Q-functions in term of S-functions using (2.6) and then performing S-function outer products and finally changing the list of S-functions into a list of Q-functions using (2.5). The raising operator method is very lengthy and cumbersome. In section 2 and 3 an easily computable combinatorial analogue of the equations (2.5) and (2.6) are given.

Evaluation of Q-function inner products without using character tables was a mystery till 1981 when Luan and Wybourne [8] used the branching $O_n \downarrow S_n$ for this purpose. Although this technique is not very efficient, the results are in an n-independent form. A more direct method for the calculation of Q-function Kronecker products is explained in section 3.

4.2 Expansion of a Q-function

A Q-function can be expanded in terms of generalised S-functions $S_\lambda(-1)$ using the inverse raising operators as given by (2.6). A large number of non-standard partitions are generated in this process which are standardised using the modification rules.

We can expand a Q-function in a more efficient way by making use of the following expression:

$$Q_\mu = \sum_\lambda b_\mu^\lambda S_\lambda, \tag{4.1}$$

where the coefficients b_μ^λ are evaluated as follows:

$$b_\mu^\lambda = \sum_{i=2}^{l(\mu)} \sum_{n_i=p}^{q} [C(t) - C(s)\delta_i](-1)^x, \tag{4.2}$$

where $q = \sum_{k=i}^{l(\mu)}(\mu_k - \lambda_k)$, $l(\mu)$ is the length of the partition (μ), p is the maximum of 0 and $\mu_i - \lambda_i$, $t = \lambda_1 - \mu_1 - n_2$, $s = \lambda_2 - \mu_2 - n_2 - 1$, $x = \sum_{j=2}^{l(\mu)} n_j$ and δ_i is equal to 1 if $\sum_{j=i+1}^{l(\mu)} \mu_j > \lambda_i$ and zero otherwise. $C(z)$ is a binomial coefficient defined as

$$C(z) = \sum_\rho \prod_{k=2}^{l(\rho)} \begin{pmatrix} N' - k + 1 \\ N(\rho_k) \end{pmatrix},$$

where N' is the number of i values such that $n_i \geq \rho_k$, $N(\rho_k)$ is the number of ρ_k's of equal weight, ρ is a partition of z such that $\sum_i^{l(\rho)} \rho_i \leq \sum_i^{l(\rho)} n_i$ and $C(z) = 0$ for $z < 0$.

The above conjecture is specifically designed to calculate the coefficients of the S-functions appearing in the expansion of a Q-function as an alternative way to using the inverse raising operators. Eq.(4.2) is suggested by the action of inverse raising operators and the modification rules for the S-functions. It has been suggested and tested by comparing the results of implementing both formulae (4.2) and the inverse raising operators on the computer. A large number of cases for the partitions μ of length $l(\mu) \leq 5$ and weight $|\mu| \leq 20$ have been tested. It is very difficult for a computer to handle the situation arising from the action of inverse raising operators on a partition μ of length $l(\mu) > 5$, whereas the algorithm of (4.2) can easily calculate the coefficient of any partition appearing in the expansion of a Q-function corresponding to the partition μ of length $l(\mu) \leq 9$.

A simple example of $\lambda \equiv 41$ and $\mu \equiv 32$ will illustrate the working of (4.2).
$i = 2$

$\mu_2 - \lambda_2 = 1, p = 1, n_2 = 1$ and $q = 1$.
$t = 4 - 3 - 1 = 0$ and $s = 1 - 2 - 1 - 1 = -3$.
Then $\rho \equiv 1$, $N' = 1$ and $N(\rho_{k=2}) = 1$.
Hence $C(t) = 1$, $C(s) = 0$, $\delta_i = 0$ and $x = 1$.
Thus $b_{32}^{41} = (1 - 0)(-1) = -1$.

Though Eq.(4.2) looks complicated, it has several advantages as stated below:

- It enables us to calculate the coefficient of a particular S-function without making the whole expansion. This simplification is very useful in the calculation of the inner product of Q-functions.
- Application of the modification rules on a rather large list of partitions is avoided.
- Its computer algorithm is extremly efficient for partitions of length greater than 3 where the inverse raising operators literally produce millions of dead and non-standard partitions.

We can easily set the highest and the lowest partitions arising in (4.1).

Definition 13 *A partition $\nu = (\nu_1\nu_2\cdots\nu_i)$ is lower than $\mu = (\mu_1\mu_2\cdots\mu_j)$ if for all $1 \leq k \leq j, \sum_k \nu_k \leq \sum_k \mu_k$ and $|\mu| = |\nu|$.*

This definition is different from Macdonald's [18] definition of lower partition and is more appropriate in this case.

Theorem 4 *The highest partition $\{\lambda\}$ appearing in the expansion of a Q-function Q_μ in terms of S-functions is $\{n\}$, where n is the weight of the partition (μ).*

Proof

Equation (4.2) and the properties of inverse raising operators observed in the previous sections lead to the above conclusion.

Corollary 3 *The coefficient*

$$b_\mu^n = \begin{cases} -1 & \text{for odd } \sum_{i=2}^{l(\mu)}(i-1)\mu_i \\ +1 & \text{for even } \sum_{i=2}^{l(\mu)}(i-1)\mu_i \end{cases} \tag{4.3}$$

Proof

The above result can be concluded from (4.2). It appears from Eq. (4.2) that for $\lambda \equiv n$, the term $[C(t) - C(s)\delta_i]$ vanishes except for $i = l(\mu)$. In that case it is 1 and $x = \sum_{i=2}^{l(\mu)}(i-1)\mu_i$.

Theorem 5 *The lowest partition $\{\lambda\}$ in the expansion of a Q-function Q_μ in terms of S-functions is $\lambda \equiv \mu$.*

Proof

It is obvious from the nature of inverse raising operators.

Corollary 4 *The coefficient*

$$b_\mu^\mu = 1.$$

Proof

Since every $\lambda_i - \mu_i$ is 0, the equation (4.2) immediately gives the above result.

This method is far more efficient and powerful than the inverse raising operators. A great advantage of this method is that the coefficient b_μ^λ of a particular λ can easily be calculated without making the whole expansion.

4.3 Expansion of an S-function

A generalised S-function $S_\lambda(-1)$ can be expanded in terms of Q-functions using the raising operators as given by (2.5). This is a very cumbersome method and generates a large number of non standard partitions which are standardised at the end using modification rules.

We can write (2.5) as follows:

$$S_\mu = \sum_\lambda g_{\lambda\mu} Q_\lambda, \tag{4.4}$$

where $g_{\lambda\mu}$ is the number of shifted tableaux of unshifted shape μ and content λ such that

1. $w = w(S)$ satisfies the shifted lattice property,
2. The leftmost i of $|w|$ is unmarked in w for $1 \leq i \leq l(\lambda)$.

The coefficients $g_{\lambda\mu}$ are easily computable using the techniques developed in [17]. Stembridge [13] has used the same coefficients in the product of a basic spin representation and an ordinary irrep of S_n.

We can set the highest and the lowest partitions arising in (4.4) using the properties of shifted tableaux of unshifted shape.

Definition 14 *The rank of a partition (ρ) is the maximum value of i for which $\rho_i \geq i$.*

Theorem 6 *The highest partition (λ) in the expansion of an S-function S_μ in terms of Q-functions is*

$$\lambda_i = \mu_i + \tilde{\mu}_i - 2i + 1 \quad for \ 1 \leq i \leq r(\mu), \tag{4.5}$$

where $\tilde{\mu}$ is the conjugate of μ and $r(\mu)$ is the rank of μ.

Proof

By theorem 1 of [17] and the fact that only the leftmost 1 of the first row can be marked, we can not make a second entry of 1 in any other row. Hence the maximum number of 1's can be placed in the first row and the first column only. Similarly maximum number of 2's can be placed in the remaining places of the second row and the second column and so on.

Corollary 5 *The highest partition (λ) in the expansion of an S-function S_μ in terms of Q-functions is the same as for $S_{\tilde{\mu}}$.*

Proof

It can easily be concluded by conjugating both sides of (4.5).

Corollary 6 *For the highest partition (λ), the coefficient*

$$g_{\lambda\mu} = 1.$$

Proof

It is clear from the proof of theorem 3 that there is only one possible tableau for the highest partition.

In order to work out the lowest (λ) in (4.4) we note that if $\mu \in \mathbf{DP}$ then the lowest $\lambda \equiv \mu$ otherwise if $\tilde{\mu} \in \mathbf{DP}$ then the lowest $\lambda \equiv \tilde{\mu}$. If both μ and $\tilde{\mu}$ have repeated parts then we can use the following algorithm.

Algorithm 2

1. If $\mu_1 < l(\mu)$ then $\lambda \equiv \tilde{\mu}$ otherwise $\lambda \equiv \mu$.
2. Reading from right to left for any $\lambda_{i+1} \geq \lambda_i$ interchange them such that
$$\lambda_1, \ldots\lambda_i, \lambda_{i+1}, \ldots\lambda_n \to \lambda_1, \ldots\lambda_{i+1}+1, \lambda_i - 1, \ldots\lambda_n.$$
3. Repeat step 2 till a partition of distinct parts is obtained.

As an example, for $\mu \equiv 433111$
$$\lambda \equiv \tilde{\mu} \equiv 6331,$$
using step 2
$$\lambda \equiv 6421.$$

4.4 Inner Products of Q-functions

The inner product of two Q-functions, $Q_\mu \circ Q_\nu$ may be written as follows [14]:
$$Q_\mu \circ Q_\nu = \sum_\lambda b^\lambda_{\mu\nu} S_\lambda, \tag{4.6}$$

where $|\mu| = |\nu| = |\lambda|$ and S_λ is an S-function. The same coefficients $b^\lambda_{\mu\nu}$ appear in the expansion of the inner product of a Q-function , Q_μ, and an S-function S_λ [14], i.e.,
$$S_\lambda \circ Q_\mu = \sum_\nu b^\lambda_{\mu\nu} 2^{-l(\nu)} Q_\nu. \tag{4.7}$$

There is no direct method of calculating the coefficients $b^\lambda_{\mu\nu}$. We can however calculate the coefficients $b^\lambda_{\mu\nu}$ in (4.7) by the following algorithm.

Algorithm 3

1. Expand Q_μ in terms of S-functions using equation (4.1).
2. Compute the S-function inner product.
3. Convert the resulting list of S-functions into Q-functions using the equation (4.4).
4. Multiply the coefficients of Q_ν by $2^{l(\nu)}$ to get $b^\lambda_{\mu\nu}$.

Instead of making the whole expansion of S-functions in terms of Q-functions in step 3 we use a direct method of computing the coefficients $b^{\lambda}_{\mu\nu}$ for any set of λ, μ and ν. For this purpose we use the following analogy.

Stembridge [13] has defined the Kronecker product of a basic spin representation $\zeta^{[\Delta;0]}$ and an ordinary irrep $\chi^{\{\mu\}}$ of S_n as follows. Let $\lambda \in \mathbf{DP}$ and $|\mu| = n$ then

$$\zeta^{[\Delta;0]} \circ \chi^{\{\mu\}} = \sum_{\lambda} \frac{1}{\varepsilon_\lambda \varepsilon_n} 2^{[l(\lambda)-1]/2} g_{\lambda\mu} \zeta^{[\Delta;\lambda]},$$

where ε_λ and ε_n are $\sqrt{2}$ or 1 according as $n - l(\lambda)$ and n are even and odd, respectively.

The coefficients $g_{\lambda\mu}$ can easily be calculated using the techniques developed earlier. For example,

$$S_{432} \circ Q_{531} = 4Q_9 + 12Q_{81} + 28Q_{72} + 38Q_{621} + 35Q_{63} + 48Q_{531} + 14Q_{432} + 17Q_{54}$$

$$\begin{aligned} Q_{432} \circ Q_{531} = \; & 2\{81\} + 4\{72\} + 4\{711\} + 4\{63\} + 12\{621\} + 6\{6111\} \\ & + 4\{54\} + 14\{531\} + 10\{522\} + 18\{5211\} + 8\{51111\} \\ & + 6\{441\} + 14\{432\} + 18\{4311\} + 18\{4221\} + 18\{42111\} \\ & + 6\{411111\} + 2\{333\} + 14\{3321\} + 10\{33111\} + 6\{3222\} \\ & + 14\{32211\} + 12\{321111\} + 4\{3111111\} + 4\{22221\} \\ & + 4\{222111\} + 4\{2211111\} + 2\{21111111\}. \end{aligned}$$

Chapter 5

Branching rules and the spin irreps of S_n

5.1 Introduction

The connection between Q-functions and S_n spin irreps is clear from equation (2.4). Using the properties and techniques studied earlier we will complete the branching rule $O_n \downarrow S_n$ for tensor and spin irreps. We will also give a simple technique of resolving Kronecker products of S_n spin irreps without making use of character tables.

5.2 The irreps of the orthogonal group O_n

The group of all the $n \times n$ orthogonal matrices for $n = 2m$ or $n = 2m + 1$ is called orthogonal group of degree n and denoted as O_n [11]. In order to define the characters of O_n, let $f(x) = 0$ be the characteristic equation of an orthogonal matrix M. If

$$1/f(x) = 1 + \sum a_r x^r,$$

then a_r is a simple or compound character of the orthogonal group.
Let

$$\frac{1-x^2}{f(x)} = 1 + \sum s_r x^r,$$

so that

$$\prod \left(\frac{1-x_i^2}{f_i} \right) = 1 + \sum s_{\mu_1} s_{\mu_2} \cdots s_{\mu_m} x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m}.$$

Then

Definition 15 *Corresponding to every partition λ into not more than m parts, there is a representation of the orthogonal group with character*

$$\chi^\lambda = \left| s_{\lambda_l - l+1}, s_{\lambda_l - l} + s_{\lambda_l - l+2}, \cdots, s_{\lambda_l - l - m+2} + s_{\lambda_l - l+m} \right|.$$

The basic spin irreps of the orthogonal group are given by

Definition 16 *The characters of the basic spin irreps Δ of degree 2^m of the orthogonal group of degree $n = 2m$ or $n = 2m+1$ are given by*

$$\chi(\Delta) = \prod \left(2\cos\frac{1}{2}\phi_r\right),$$

except in the case of a matrix element of negative determinant for $n = 2m$, in which case

$$\chi(\Delta) = 0.$$

The other spin characters are calculated by noting the fact that the direct product of a spin representation and an ordinary representation will give a spin representation of the orthogonal group. A detailed account can be found in [11].

5.3 The irreps of the symmetric group S_n

All the $n \times n$ permutation matrices form a group called symmetric group denoted by S_n. The different methods of calculation of the characters can be found in literature [11, 6, 7, 23]. A very useful way of writing the irreducible character of the irrep $[\lambda]$ corresponding to the partition λ for the class ρ, is:

$$s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^{[\lambda]} p_\rho,$$

where s_λ is an S-function corresponding to the partition λ.

The symmetric group S_n of order $n!$ has two spin groups Γ_n and Γ'_n of order $2(n!)$ [9]. The characters of the two groups are trivially related. The characters of the positive classes of Γ_n and Γ'_n are the same, whilst the characters of the negative classes of Γ'_n can be found by multiplying the corresponding characters of Γ_n by -1.
The basic spin characters of the symmetric group are defined as [9].

Definition 17 *The basic spin character of the class $(\rho) = (1^{\alpha_1}3^{\alpha_3}5^{\alpha_5}\cdots)$ of Γ_n is $2^{(p-1-\epsilon)/2}$, where $\epsilon = 0$ or 1 according as n is odd or even, and p denotes the total number of cycles in (ρ) and of the class (n), when n is even, is $i^{n/2}\sqrt{n/2}$. The basic spin character of all other classes is zero.*

The remaining spin characters can be calculated by making use of (2.4), originally given by Schur [1].

5.4 Branching

Given a group, G, with elements $\{g, ...\}$ and irreps $\{\lambda_G, ...\}$, and a finite subgroup, H, with elements $\{h, ...\}$ and irreps $\{\mu_H, ...\}$, then the restriction of the set of matrices $\{\lambda_G(g)\}$, forming the representation $[\lambda]_G$ of G, to the set $\{\lambda_G(h)\}$ yields a representation of H which is in general, reducible. If $\forall\ h \in H$

$$\lambda_G(h) = \sum_{\mu_H} a_{\lambda_G}^{\mu_H} \mu_H(h),$$

then it is said that under the restriction $G \downarrow H$ the representation $[\lambda]_G$ reduces into a set of irreps any one of which, $[\mu]_H$, occurs with multiplicity $a^{\mu_H}_{\lambda_G}$.

The coefficients $a^{\mu_H}_{\lambda_G}$ can be calculated using the character tables, such that

$$a^{\mu_H}_{\lambda_G} = \frac{1}{n} \sum_{\rho} n_\rho \chi^{[\lambda]_G}_{\rho} \chi^{[\mu]_H}_{\rho},$$

where n is the order of the subgroup H and n_ρ is the order of the class (ρ).

This method is very tedious, on the other hand, the theory of symmetric functions provides a very powerful tool for this purpose, which will be discussed later in this chapter.

5.5 Labelling of the Irreps of O_n and S_n

The tensor irreps of O_n are usually labelled by $[\lambda]$ and the spin irreps by $[\Delta; \lambda]$ where $\Delta \equiv [\Delta; 0]$ is the basic spin irrep. The tensor irreps of S_n are labelled by $\{\lambda\}$ and can be written as $\{n-m, \mu\}$ where (μ) is a partition of m. In reduced notation $\{\lambda\}$ may then be written as $<\mu>$ and the spin irreps as $\{\Delta; \lambda\}$ and in reduced notation $<\Delta; \mu>$ [10].

If $(n-k)$ is even then the spin irrep of S_n is self- associated, otherwise it will split into an associated pair of spin irreps designated as $\{\Delta; \lambda\}_+$ and $\{\Delta; \lambda\}_-$, where k is the length, $l(\lambda)$, of the partition λ. For n even the basic spin irrep Δ will split into an associated pair Δ_+ and Δ_-.

The dimension formula for the spin irreps of S_n has been given by Schur [1]. We modify it for reduced notation as follows

$$f_n^{<\Delta;\mu>} = 2^{[(n-r-1)/2]} \prod_{x=0}^{m-1}(n-x) \prod_{i=1}^{r} \frac{1}{\mu_i!} \left(\frac{n-m-\mu_i}{n-m+\mu_i}\right) \prod_{0<i<s\leq r} \left(\frac{\mu_i-\mu_s}{\mu_i+\mu_s}\right), \quad (5.1)$$

where m is the weight and r the number of parts of the partition (μ).

5.6 $O_n \downarrow S_n$ Branching Rule for Tensor Irreps

S_n is treated as a subgroup of O_n. Consider the embedding

$$[1] \downarrow <1> + <0>.$$

One can decompose an ordinary irrep $[\lambda]$ of O_n into irreps of S_n using the inner plethysm

$$(<1> + <0>) \otimes [\lambda].$$

Techniques developed by King [15] readily lead to the result

$$[\lambda] \downarrow <1> \otimes \{\lambda/G,\} \quad (5.2)$$

where

$$G = \sum_{\varepsilon}(-1)^{(e-r)/2}\{\varepsilon\},$$

where ε is a self-conjugate partition of weight e and of rank r.

Plethysms of the type $< 1 > \otimes \{\mu\}$ may be evaluated by noting that any S-function $\{\mu\}$ may be expanded as the product of S-functions of the type $\{1^x\}$ such that $\{1^x\} = a_x$ where a_x is an elementary symmetric function $\alpha_1\alpha_2...\alpha_x$ and

$$\{\mu\} = | a_{\tilde{\mu}_s - s + t} |,$$

where $\tilde{\mu}$ is the partition conjugate to μ and by using the identity

$$< 1 > \otimes \{1^r\} = < 1^r > .$$

For example, an irrep [321] of O_n is branched to S_n in reduced notation as

$$\begin{aligned}[321] \quad \rightarrow \quad & < 41 > + 2 < 4 > + < 321 > + 3 < 32 > + 3 < 311 > \\ & + 9 < 31 > + 6 < 3 > + 3 < 221 > + 7 < 22 > \\ & + < 2111 > + 9 < 211 > + 15 < 21 > + 6 < 2 > \\ & + 2 < 1111 > + 6 < 111 > + 6 < 11 > + 2 < 1 > . \end{aligned} \tag{5.3}$$

The above calculation is laborious but was readily evaluated using the program SCHUR which rapidly evaluates reduced products. For a particular value of n we may convert from the reduced notation to S-functions of weight n and the resulting list of S-functions can be standarised using the modification rules given by Littlewood [11].

5.7 $O_n \downarrow S_n$ Branching Rule for Spin Irreps

A spin irrep of O_n can be written as the product of the basic spin irrep and ordinary irreps as follows:

$$[\Delta; \lambda] = \Delta \cdot [\lambda/P], \tag{5.4}$$

where P is the S-function series [15],

$$P = \sum_m (-1)^m \{m\},$$

and (m) is a partition of one part only.

The ordinary irreps of O_n can be decomposed into ordinary irreps of S_n using equation (5.2). Hence

$$[\Delta; \lambda] \downarrow \Delta \cdot < 1 > \otimes \{\lambda/PG\} \equiv \Delta \cdot < 1 > \otimes \{\lambda/A\}, \tag{5.5}$$

where A is the S-function series [15],

$$A = \sum_\alpha (-1)^{a/2} \{\alpha\},$$

where a is the weight of the partition and (α) are partitions in Frobenius notation such that

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix}.$$

The basic spin representation Δ of O_n is branched into the basic spin representation Δ of S_n as $\Delta \downarrow \Delta$. Finally the inner product of the basic spin representation and tensor irreps of S_n is performed using the Young raising operator [10].

Thus the algorithm for evaluating the branching rule $O_n \downarrow S_n$ for spin irreps $[\Delta;\lambda]$ of O_n is as follows.

Algorithm 4

1. Evaluate the terms, in reduced notation, contained in

$$<1> \otimes \{\lambda/A\}.$$

2. Apply the raising operator $\prod(1+\delta_{ij})$ to the terms produced in (1) and standarise the resulting partitions as Q-functions and then replace each Q-function $Q_{<\lambda>}$ by $<\Delta;\lambda>$.

3. For a particular value of n multiply each $<\Delta;\lambda>$ by $2^{[(l(\lambda)-n(mod 2))/2]}$ and replace $<\Delta;\lambda>$ by $\{\Delta;\lambda\}$ provided that $\{\Delta;\lambda\}$ splits into $\{\Delta;\lambda\}_+$ and $\{\Delta;\lambda\}_-$ if $(n-l(\lambda))$ is odd.

As an example,

$$\begin{aligned}
[\Delta;432] \quad \rightarrow \quad & 2<\Delta;71> + 10<\Delta;7> + <\Delta;63> + <\Delta;621> \\
& + 14<\Delta;62> + 47<\Delta;61> + 93<\Delta;6> \\
& + <\Delta;54> + 2<\Delta;531> + 22<\Delta;53> + 14<\Delta;521> \\
& + 118<\Delta;52> + 234<\Delta;51> + 312<\Delta;5> \\
& + <\Delta;432> + 13<\Delta;431> + 76<\Delta;43> + 56<\Delta;421> \\
& + 328<\Delta;42> + 506<\Delta;41> + 544<\Delta;4> \\
& + 71<\Delta;321> + 350<\Delta;32> + 556<\Delta;31> \\
& + 560<\Delta;3> + 282<\Delta;21> + 350<\Delta;2> \\
& + 124<\Delta;1> + 38<\Delta;0>.
\end{aligned}$$

This is a universal list in the sense that once it is calculated it holds for all n though, for small n modification rules may need to be applied. Using step (3) of the above algorithm one can specialise the list to a particular value of n.

5.8 Kronecker Products of S_n Spin Irreps

The inner product of an ordinary irrep $\{\lambda\}$ and a spin irrep of $[\Delta;\mu]$ can easily be worked out using (4.7) and the *Algorithm 2* [17]. In order to convert the Q-functions into representations, the right hand side of the equation (4.7) needs to be multiplied by $2^{[l(\nu)-n(mod2)+1)/2]}$ and divided by $2^{[l(\mu)-n(mod2)+1)/2]}$ where $[x]$ means only the integer part of x. We also observe that

$$[\Delta;\mu]_\pm \otimes \{\lambda\} = 1/2([\Delta;\mu]\otimes\{\lambda\}). \tag{5.6}$$

If $(n-k)$ is odd then $[\Delta;\lambda]$ splits into an associate pair as follows:

$$[\Delta;\lambda] = [\Delta;\lambda]_+ + [\Delta;\lambda]_-. \tag{5.7}$$

Then we need difference characters for a complete resolution of Kronecker products of the type $[\Delta;\lambda]^2_\pm$ or $[\Delta;\lambda]_\pm[\Delta;\lambda]_\mp$.

The difference character $[\Delta;\lambda]'$ is defined as

$$[\Delta;\lambda]' = [\Delta;\lambda]_+ - [\Delta;\lambda]_-. \tag{5.8}$$

From (5.7) we get

$$[\Delta;\lambda]^2 = [\Delta;\lambda]_+[\Delta;\lambda]_+ + [\Delta;\lambda]_-[\Delta;\lambda]_- + 2[\Delta;\lambda]_+[\Delta;\lambda]_-, \tag{5.9}$$

and from (5.8) we get

$$[\Delta;\lambda]'^2 = [\Delta;\lambda]_+[\Delta;\lambda]_+ + [\Delta;\lambda]_-[\Delta;\lambda]_- - 2[\Delta;\lambda]_+[\Delta;\lambda]_-. \tag{5.10}$$

Adding (5.9) and (5.10)

$$[\Delta;\lambda]_\pm[\Delta;\lambda]_\pm = 1/4([\Delta;\lambda]^2 + [\Delta;\lambda]'^2), \tag{5.11}$$

and subtracting (5.10) from (5.9)

$$[\Delta;\lambda]_\pm[\Delta;\lambda]_\mp = 1/4([\Delta;\lambda]^2 - [\Delta;\lambda]'^2), \tag{5.12}$$

where

$$[\Delta;\lambda]_+[\Delta;\lambda]_+ = [\Delta;\lambda]_-[\Delta;\lambda]_-,$$

and

$$[\Delta;\lambda]_+[\Delta;\lambda]_- = [\Delta;\lambda]_-[\Delta;\lambda]_+.$$

For two different spin irreps of S_n say $[\Delta;\mu]$ and $[\Delta;\nu]$ the equations analogous to (5.11) and (5.12) reduce to

$$[\Delta;\mu]_\pm[\Delta;\nu]_\pm = [\Delta;\mu]_\pm[\Delta;\nu]_\mp = 1/4[\Delta;\mu][\Delta;\nu], \tag{5.13}$$

since

$$[\Delta;\mu]'[\Delta;\nu]' = 0 \qquad \text{for} \quad \mu \neq \nu.$$

For any μ and ν including $\mu = \nu$ we get the following

$$[\Delta;\mu]_{\pm}[\Delta;\nu] = 1/2[\Delta;\mu][\Delta;\nu] \tag{5.14}$$

$[\Delta;\mu][\Delta;\nu]$ or $[\Delta;\lambda]^2$ can be calculated using the *Algorithm 4*.

For the calculation of difference characters let us consider $[\Delta;\lambda]'^2$. This will have non-zero characters only for the classes of (λ) and $(\Delta;\lambda)$. We can expand $[\Delta;\lambda]'^2$ in terms of ordinary irreps $\{\rho\}$ of S_n as follows:

$$[\Delta;\lambda]'^2 = g^{\rho}_{\lambda\lambda}\{\rho\}$$

where $g^{\rho}_{\lambda\lambda} = 2i^{n-k+1}\chi^{\rho}_{\lambda}$.

The character χ^{ρ}_{λ} can easily be calculated using Littlewood's theorem [11].

Some typical examples of products for the symmetric group S_{10} that illustrate the above results are given below.

$$\begin{aligned}
[\Delta;32]\otimes[\Delta;31] \quad \to \quad & 4\{91\}+16\{82\}+16\{811\}+32\{73\} \\
& +68\{721\}+36\{7111\}+36\{64\}+124\{631\} \\
& +88\{622\}+140\{6211\}+52\{61111\}+16\{55\} \\
& +108\{541\}+168\{532\}+216\{5311\}+200\{5221\} \\
& +176\{52111\}+52\{511111\}+92\{442\} \\
& +112\{4411\}+76\{433\}+284\{4321\}+200\{43111\} \\
& +112\{4222\}+216\{42211\}+140\{421111\} \\
& +36\{4111111\}+76\{3331\}+92\{3322\} \\
& +168\{33211\}+88\{331111\}+108\{32221\}+124\{322111\} \\
& +68\{3211111\}+16\{31111111\}+16\{22222\}+36\{222211\} \\
& +32\{2221111\}+16\{22111111\}+4\{211111111\},
\end{aligned}$$

$$\begin{aligned}
[\Delta;32]_{\pm}\otimes[\Delta;31] \quad \to \quad & 2\{91\}+8\{82\}+8\{811\}+16\{73\} \\
& +34\{721\}+18\{7111\}+18\{64\}+62\{631\} \\
& +44\{622\}+70\{6211\}+26\{61111\}+8\{55\} \\
& +54\{541\}+84\{532\}+108\{5311\}+100\{5221\} \\
& +88\{52111\}+26\{511111\}+46\{442\} \\
& +56\{4411\}+38\{433\}+142\{4321\}+100\{43111\} \\
& +56\{4222\}+108\{42211\}+70\{421111\} \\
& +18\{4111111\}+38\{3331\}+46\{3322\} \\
& +84\{33211\}+44\{331111\}+54\{32221\}+62\{322111\} \\
& +34\{3211111\}+8\{31111111\}+8\{22222\}+18\{222211\} \\
& +16\{2221111\}+8\{22111111\}+2\{211111111\},
\end{aligned}$$

$$
\begin{aligned}
[\Delta;32]_\pm \otimes [\Delta;32]_\pm \quad \to \quad & \{10\} + \{91\} + 4\{82\} + 4\{811\} + 5\{73\} \\
& +11\{721\} + 7\{7111\} + 4\{64\} + 18\{631\} \\
& +13\{622\} + 21\{6211\} + 8\{61111\} + 3\{55\} \\
& +13\{541\} + 22\{532\} + 29\{5311\} + 27\{5221\} \\
& +25\{52111\} + 9\{511111\} + 12\{442\} + 14\{4411\} \\
& +9\{433\} + 36\{4321\} + 27\{43111\} + 14\{4222\} \\
& +29\{42211\} + 21\{421111\} + 6\{4111111\} + 10\{3331\} \\
& +11\{3322\} + 22\{33211\} + 13\{331111\} + 14\{32221\} \\
& +17\{322111\} + 12\{3211111\} + 4\{31111111\} + \{22222\} \\
& +5\{222211\} + 5\{2221111\} + 3\{22111111\} + 2\{211111111\},
\end{aligned}
$$

$$
\begin{aligned}
[\Delta;32]_\pm \otimes [\Delta;32]_\mp \quad \to \quad & 2\{91\} + 3\{82\} + 4\{811\} + 5\{73\} + 12\{721\} \\
& +6\{7111\} + 5\{64\} + 17\{631\} + 13\{622\} \\
& +21\{6211\} + 9\{61111\} + \{55\} + 14\{541\} \\
& +22\{532\} + 29\{5311\} + 27\{5221\} + 25\{52111\} \\
& +8\{511111\} + 11\{442\} + 14\{4411\} + 10\{433\} \\
& +36\{4321\} + 27\{43111\} + 14\{4222\} + 29\{42211\} \\
& +21\{421111\} + 7\{4111111\} + 9\{3331\} + 12\{3322\} \\
& +22\{33211\} + 13\{331111\} + 13\{32221\} \\
& +18\{322111\} + 11\{3211111\} + 4\{31111111\} + 3\{22222\} \\
& +4\{222211\} + 5\{2221111\} + 4\{22111111\} \\
& +\{211111111\} + \{1111111111\},
\end{aligned}
$$

$$
\begin{aligned}
[\Delta;32]_\pm \otimes \{532\} \quad \to \quad & 23[\Delta;41]_+ + 23[\Delta;41]_- + 36[\Delta;4] + 5[\Delta;321] \\
& +22[\Delta;32]_+ + 22[\Delta;32]_- + 42[\Delta;31]_+ + 42[\Delta;31]_- \\
& +43[\Delta;3] + 22[\Delta;21]_+ + 22[\Delta;21]_- + 26[\Delta;2] \\
& +8[\Delta;1] + [\Delta;0]_+ + [\Delta;0]_-,
\end{aligned}
$$

$$
\begin{aligned}
[\Delta;32] \otimes \{532\} \quad \to \quad & 46[\Delta;41]_+ + 46[\Delta;41]_- + 72[\Delta;4] + 10[\Delta;321] \\
& +44[\Delta;32]_+ + 44[\Delta;32]_- + 84[\Delta;31]_+ + 84[\Delta;31]_- \\
& +86[\Delta;3] + 44[\Delta;21]_+ + 44[\Delta;21]_- + 52[\Delta;2] \\
& +16[\Delta;1] + 2[\Delta;0]_+ + 2[\Delta;0]_-.
\end{aligned}
$$

Chapter 6

The $q-$deformation of symmetric functions

6.1 Introduction

The quantum groups, recently introduced by Drinfeld [22] and Woronowicz [45], which are not groups in proper sense, but are the q-deformation of Lie algebras, provide a new mathematical tool for the study of quantum field theory. The q-deformation of Kac-Moody algebras or quantum affine algebras were first introduced by Drinfeld [22] and Jimbo [38]. The q-analogue of a Kac-Moody algebra may be defined for any generalized symmetrizable Cartan matrix. A detail account of this technique will be discussed in the next chapter in connection with the vertex operator algebra.

In recent years the q-analogues of hypergeometric polynomial functions have also been used in the description of quantum groups, for example the q-Jacobi polynomials appear as the matrix elements of the irreducible representations of the quantum group $SU_q(2)$ and q-Hahn polynomials admit a similar quantum group theoretic interpretation.

In a very recent paper King and Wybourne [40] have presented an S-function approach to the calculation of the characters of the Hecke algebras $H_n(q)$ of type A_{n-1}. In a sense it is the first time that the q-deformed characters have been explicitly calculated. It is important to note that for the limit $q \to 1$, the generators g_i of $H_n(q)$ can be replaced by s_i, where s_i is a transposition $(i, i+1)$. It has also been shown that there exists a map $h : S_n \to H_n(q)$ such that $h(s_i) = g_i$.

In chapter 2 we introduced the basic ideas and definitions related to symmetric functions. Here we will discuss the q-deformation of symmetric functions and will give a q -analogue of the ordinary and spin characters of the symmetric group S_n. The methods used closely parallel those of quantum groups.

The content of this chapter is as follows. In section 6.2 we give a brief account of different types of q-deformations. Section 6.3 is devoted to the description of classical symmetric functions with certain examples. Section 6.4 deals with the q-deformation of the classical symmetric functions with explicitly calculated examples. A q-deformation of the Hall-Littlewood symmetric functions is given in section 6.5. As a consequence of this deformation, a q-analogue of the symmetric group is presented in the last section. The q-deformed characters of the symmetric group is calculated explicitly. A q-analogue of the Kostka-Foulkes polynomials, which play a very important role in the form of transition matrices in the theory of symmetric functions is also given in the same section.

6.2 q-Deformation

There are different types of q-deformations used in the combinatoric and group theoretic literature. Here we will give a brief account of them.

Drinfeld-Jimbo type q-deformation

In [38] Jimbo has used the following q-deformed commutation relations among the generators $\eta, \phi, \theta$of a Lie algebra,

$$[\eta, \phi] = 2\phi, \quad [\eta, \theta] = -2\theta, \quad [\phi, \theta] = \sinh(q\eta)/\sinh(q),$$

where q is a parameter. Drinfeld [22] has also used a similar form of q-deformation. In some combinatorial literature, for example Biedenharn [39], the following definition of a q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

has been used whereas Jing and Frenkel use the following:

$$[n]_q = \frac{q^n - q^{-n}}{t},$$

where $q = e^{t/2}$.

q- Hypergeometric Functions

The following q-deformations of Jacobi and Hahn polynomials are given in the description representations of the twisted $SU(2)$ quantum group in [41].

For $r \in \mathcal{Z}_+$ the q-hypergeometric series ${}^{r+1}\phi_r$ is defined by

$${}^{r+1}\phi_r \left(\begin{array}{c} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{array} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_{r+1}; q)_k}{(b_1, \cdots, b_r; q)_k} z^k,$$

where

$$(b_1, \cdots, b_n; q)_k = (b_1; q)_k \cdots (b_n; q)_k,$$

and

$$(b; q)_k = \prod_{i=0}^{k-1} (1 - bq^i),$$

is a q-shifted factorial.

The q-Jacobi polynomials are defined in terms of ${}^2\phi_1$ such that

$$p_n(x;a,b|q) = {}^2\phi_1\left(\begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array} ; q, qx\right),$$

and the q-Hahn polynomials are defined in terms of ${}^3\phi_2$ such that

$$\mathcal{L}_n(x;a,b,N|q) = {}^3\phi_2\left(\begin{array}{c} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{array} ; q, q\right),$$

for $N \in \mathcal{Z}_+$ and $n \in \{0, 1, \cdots, N\}$.

King-Wybourne q-deformation

In a very recent paper [40] King and Wybourne have shown that the characters of the complex Hecke algebra $H_n(q)$, with q an arbitrary but fixed parameter, generated by g_i with $i = 1, 2, \cdots, n-1$ subject to the relations:

$$\mathrm{g}_i^2 = (q-1)\mathrm{g}_i + q \quad for\ i = 1, 2, \cdots, n-1,$$
$$\mathrm{g}_i\mathrm{g}_{i+1}\mathrm{g}_i = \mathrm{g}_{i+1}\mathrm{g}_i\mathrm{g}_{i+1} \quad for\ i = 1, 2, \cdots, n-2,$$
$$\mathrm{g}_i\mathrm{g}_j = \mathrm{g}_j\mathrm{g}_i \quad |i-j| \geq 2,$$

can be calculated in terms of the properties of Schur-functions and certain $q-$deformed power sum symmetric functions. Their analysis involves a $q-$deformation in terms of q-numbers defined by

$$[n]_q = 1 + q + \cdots + q^{n-1}. \tag{6.1}$$

In our description of the q-deformation of the symmetric functions we will use the same definition of q-number. Although we will show in the next chapter that it is possible to derive consistently similar expressions using any other type of q-deformation.

6.3 Classical Symmetric Functions

In chapter 2 we have given a brief account of the ring of symmetric functions Λ_Q. The bases of Λ_Q are called *classical symmetric functions.* In this section we will briefly review them. For details we refer [18, 20].

6.3.1 Monomial Symmetric Functions

Using the notations introduced in chapter 2 we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}.$$

Let $\lambda \vdash n$. The polynomial

$$m_\lambda(x_1, \ldots, x_n) = \sum x^\alpha, \tag{6.2}$$

summed over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_n)$, is a symmetric function, the so-called *monomial symmetric function.*

6.3.2 Elementary Symmetric Functions

For each integer $r \geq 0$ the rth *elementary symmetric function* e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} = m_{(1^r)}, \tag{6.3}$$

for $r \geq 1$. The generating function for the e_r is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t). \tag{6.4}$$

For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots. \tag{6.5}$$

6.3.3 Complete Symmetric Functions

For each $r \geq 0$ the rth *complete symmetric function* h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \ldots$, so that

$$h_r = \sum_{|\lambda| = r} m_\lambda. \tag{6.6}$$

We have $h_0 = 1$ and $h_1 = e_1$. We define h_r and e_r to be zero for $r < 0$.

The generating function for the h_r is

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}. \tag{6.7}$$

Note that $H(t)E(-t) = 1$ and hence

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0, \tag{6.8}$$

for all $n \geq 1$. Again we may define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots. \tag{6.9}$$

6.3.4 Power Sum Symmetric Functions

For each $r \geq 1$ the rth *power sum* is;

$$p_r = \sum x_i^r = m_{(r)}. \tag{6.10}$$

The generating function for the p_r is;

$$\begin{aligned} P(t) &= \sum_{r\geq 1} p_r t^{r-1}, \\ &= \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1}, \\ &= \sum_{i\geq 1} \frac{x_i}{1-x_i t}, \\ &= \sum_{i\geq 1} \frac{d}{dt} log \frac{1}{1-x_i t}, \end{aligned} \tag{6.11}$$

leading to

$$P(t) = H'(t)/H(t), \tag{6.12}$$

and hence to

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r}, \tag{6.13}$$

for $n \geq 1$. Again we may define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots. \tag{6.14}$$

Let us write

$$z_\lambda = \prod_{i\geq 1} i^{m_i}.m_i!, \tag{6.15}$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. and may write

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda, \tag{6.16}$$

and

$$e_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda, \tag{6.17}$$

where $\varepsilon_\lambda = (-1)^{n-l(\lambda)}$.

6.3.5 Determinantal Relations

The symmetric functions described above can be expanded in terms of each other by the following determinantal relations.

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \ldots & 0 \\ 2e_2 & e_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ne_n & e_{n-1} & e_{n-2} & \ldots & e_1 \end{vmatrix} \tag{6.18}$$

$$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \ldots & 0 \\ p_2 & p_1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & . & \ldots & n-1 \\ p_n & p_{n-1} & . & \ldots & p_1 \end{vmatrix} \tag{6.19}$$

$$(-1)^{n-1}p_n = \begin{vmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ nh_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix} \tag{6.20}$$

$$n!h_n = \begin{vmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \dots & -n+1 \\ p_n & p_{n-1} & \cdot & \dots & p_1 \end{vmatrix} \tag{6.21}$$

6.3.6 Examples for Classical Symmetric Functions

Using the determinantal relations given above, some of the classical symmetric functions upto $n = 4$ is calculated as

1. $p_n \to e_\lambda$;

$$\begin{aligned} n &= 1 \quad p_1 = e_1, \\ n &= 2 \quad p_2 = e_{1^2} - 2e_2, \\ n &= 3 \quad p_3 = e_{1^3} - 3e_{21} + 3e_3, \\ n &= 4 \quad p_4 = e_{1^4} - 4e_{21^2} + 2e_{2^2} + 4e_{31} - 4e_4. \end{aligned} \tag{6.22}$$

2. $e_n \to p_\lambda$;

$$\begin{aligned} n &= 1 \quad e_1 = p_1, \\ n &= 2 \quad e_2 = \frac{p_{1^2} - p_2}{2}, \\ n &= 3 \quad e_3 = \frac{p_{1^3} - 3p_{21} + 2p_3}{6}, \\ n &= 4 \quad e_4 = \frac{p_{1^4} - 6p_{21^2} + 3p_{2^2} + 8p_{31} - 6p_4.}{24} \end{aligned} \tag{6.23}$$

3. $p_n \to h_\lambda$;

$$\begin{aligned} n &= 1 \quad p_1 = h_1, \\ n &= 2 \quad p_2 = -h_{1^2} + 2h_2, \\ n &= 3 \quad p_3 = h_{1^3} - 3h_{21} + 3h_3, \\ n &= 4 \quad p_4 = -h_{1^4} + 4h_{21^2} - 2h_{2^2} - 4h_{31} + 4h_4. \end{aligned} \tag{6.24}$$

4. $h_n \to p_\lambda$;

$$\begin{aligned} n &= 1 \quad h_1 = p_1, \\ n &= 2 \quad h_2 = \frac{p_{1^2} + p_2}{2}, \\ n &= 3 \quad h_3 = \frac{p_{1^3} + 3p_{21} + 2p_3}{6}, \\ n &= 4 \quad h_4 = \frac{p_{1^4} + 6p_{21^2} + 3p_{2^2} + 8p_{31} + 6p_4.}{24} \end{aligned} \tag{6.25}$$

6.4 $q-$deformation of Classical Symmetric Functions

Similar to the development of Hecke algebra, we introduce an arbitrary but fixed parameter q in the description of the classical symmetric functions. The classical symmetric functions so obtained are called "q-deformed classical symmetric functions". Since every symmetric function can be expanded in terms of power sum symmetric function $p_\lambda(x)$, we start with the q-deformation of $p_\lambda(x)$. The relationship between Schur functions and power sum symmetric functions is given as [40],

$$p_r(x) = \sum_{a,b=0,a+b+1=r}^{r-1} (-1)^b s_{(a+1,1^b)}(x), \tag{6.26}$$

then the q-analogue of (6.26) can be written as

$$p_r(q;x) = \sum_{a,b=0,a+b+1=r}^{r-1} (-1)^b q^a s_{(a+1,1^b)}(x), \tag{6.27}$$

and for $\rho = (\rho_1, \rho_2, \ldots)$ let

$$p_\rho(q;x) = p_{\rho_1}(q;x) p_{\rho_2}(q;x) \ldots . \tag{6.28}$$

Clearly under $q \to 1$ Eq.(6.27) goes over to Eq.(6.26).

Using the King-Wybourne q-number (6.1) we note the following special cases:

1. $q = 1$ In that case $[n]_1 = n$,
2. $q = 0$ In that case $[n]_0 = 1$,
3. $q = -1$ In that case $[n]_{-1} = n \bmod 2$.

6.4.1 $q-$deformed Determinantal Relations

We define the $q-$deformed analogues of Eqs. (6.18)-(6.21) as follows:

$$p_n = \begin{vmatrix} [1]_q e_1 & 1 & 0 & \ldots & 0 \\ [2]_q e_2 & q e_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [n]_q e_n & q^{n-1} e_{n-1} & q^{n-2} e_{n-2} & \ldots & q e_1 \end{vmatrix}, \tag{6.29}$$

$$[n]_q! e_n = \begin{vmatrix} p_1 & [1]_q & 0 & \ldots & 0 \\ p_2 & q p_1 & [2]_q & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & q p_{n-2} & . & \ldots & [n-1]_q \\ p_n & q p_{n-1} & q^2 p_{n-2} & \ldots & q^{n-1} p_1 \end{vmatrix}, \tag{6.30}$$

$$(-1)^{n-1}p_n = \begin{vmatrix} [1]_q h_1 & 1 & 0 & \dots & 0 \\ [2]_q h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [n]_q h_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix}, \tag{6.31}$$

$$[n]_q!h_n = \begin{vmatrix} p_1 & -[1]_q & 0 & \dots & 0 \\ p_2 & p_1 & -[2]_q & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & . & \dots & -[n]_q \\ p_n & p_{n-2} & . & \dots & p_1 \end{vmatrix}, \tag{6.32}$$

where the p_n are the $q-$deformed power sums defined in (6.27).

6.4.2 Some Examples of $q-$deformed Symmetric Functions

1. $p_n \to e_\lambda$;

$$\begin{aligned} n=1 \quad p_1 &= e_1, \\ n=2 \quad p_2 &= qe_{1^2} - (q+1)e_2, \\ n=3 \quad p_3 &= q^2e_{1^3} - q(1+2q)e_{21} + (1+q+q^2)e_3, \\ n=4 \quad p_4 &= q^3e_{1^4} - q^2(1+3q)e_{21^2} + q^2(1+q)e_{2^2} + q(1+q+2q^2)e_{31}, \\ & -(1+q+q^2+q^3)e_4. \end{aligned} \tag{6.33}$$

2. $e_n \to p_\lambda$;

$$\begin{aligned} n=1 \quad e_1 &= p_1, \\ n=2 \quad e_2 &= \frac{qp_{1^2}-p_2}{(1+q)}, \\ n=3 \quad e_3 &= \frac{q^3p_{1^3}-q(1+2q)p_{21}+(1+q)p_3}{(1+q)(1+q+q^2)}, \\ n=4 \quad e_4 &= \frac{q^6p_{1^4}-q^3(1+2q+3q^2)p_{21^2}+q^2(1+q+q^2)p_{2^2}+q(1+q)(1+q+2q^2)p_{31}-(1+q)(1+q+q^2)p_4}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}, \end{aligned} \tag{6.34}$$

3. $p_n \to h_\lambda$;

$$\begin{aligned} n=1 \quad p_1 &= h_1, \\ n=2 \quad p_2 &= -h_{1^2} + (1+q)h_2, \\ n=3 \quad p_3 &= h_{1^3} - (2+q)h_{21} + (1+q+q^2)h_3, \\ n=4 \quad p_4 &= -h_{1^4} + (3+q)h_{21^2} - (1+q)h_{2^2} - (2+q+q^2)h_{31} \\ & +(1+q+q^2+q^3)h_4, \end{aligned} \tag{6.35}$$

4. $h_n \to p_\lambda$;

$$n=1 \quad h_1 = p_1,$$

$$\begin{aligned}
n=2 \quad h_2 &= \frac{p_{1^2}+p_2}{(1+q)},\\
n=3 \quad h_3 &= \frac{p_{1^3}+(2+q)p_{21}+(1+q)p_3}{(1+q)(1+q+q^2)},\\
n=4 \quad h_4 &= \frac{p_{1^4}+(3+2q+q^2)p_{21^2}+(1+q+q^2)p_{2^2}+(2+3q+2q^2+q^3)p_{31}+(1+2q+2q^2+q^3)p_4}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}.
\end{aligned} \tag{6.36}$$

For $q \to 1$ the Eqs. (6.33)-(6.36) reduce to (6.22)-(6.25).

For $q = 0$ we obtain the following results:

1. $p_n \to e_\lambda$;

$$\begin{aligned}
n &= 1 \quad p_1 = e_1,\\
n &= 2 \quad p_2 = -e_2,\\
n &= 3 \quad p_3 = e_3,\\
n &= 4 \quad p_4 = -e_4.
\end{aligned} \tag{6.37}$$

2. $e_n \to p_\lambda$;

$$\begin{aligned}
n &= 1 \quad e_1 = p_1,\\
n &= 2 \quad e_2 = -p_2,\\
n &= 3 \quad e_3 = p_3,\\
n &= 4 \quad e_4 = -p_4.
\end{aligned} \tag{6.38}$$

3. $p_n \to h_\lambda$;

$$\begin{aligned}
n &= 1 \quad p_1 = h_1,\\
n &= 2 \quad p_2 = -h_{1^2} + h_2,\\
n &= 3 \quad p_3 = h_{1^3} - 2h_{21} + h_3,\\
n &= 4 \quad p_4 = -h_{1^4} + 3h_{21^2} - h_{2^2} - 2h_{31} + h_4.
\end{aligned} \tag{6.39}$$

4. $h_n \to p_\lambda$;

$$\begin{aligned}
n &= 1 \quad h_1 = p_1,\\
n &= 2 \quad h_2 = p_{1^2} + p_2,\\
n &= 3 \quad h_3 = p_{1^3} + 2p_{21} + p_3,\\
n &= 4 \quad h_4 = p_{1^4} + 3p_{21^2} + p_{2^2} + 2p_{31} + p_4.
\end{aligned} \tag{6.40}$$

6.4.3 q-Deformed monomial expansions

Using the explicitly calculated examples in (6.4.2), we conclude the following results:

$$\begin{aligned}
p_1(q;x) &= m_1(x),\\
p_2(q;x) &= qm_2(x) + (q-1)m_{1^2}(x),
\end{aligned}$$

$$\begin{aligned} p_3(q;\boldsymbol{x}) &= q^2 m_3(\boldsymbol{x}) + q(q-1)m_{21}(\boldsymbol{x}) + (q-1)^2 m_{1^3}(\boldsymbol{x}), \\ p_4(q;\boldsymbol{x}) &= q^3 m_4(\boldsymbol{x}) + q^2(q-1)[m_{31}(\boldsymbol{x}) + m_{2^2}(\boldsymbol{x})], \\ &\quad + q(q-1)^2 m_{21^2}(\boldsymbol{x}) + (q-1)^3 m_{1^4}(\boldsymbol{x}). \end{aligned} \tag{6.41}$$

In general we have

$$p_r(q;\boldsymbol{x}) = \sum_{\rho\vdash r} (q-1)^{l(\rho)-1} q^{r-l(\rho)} m_\rho(\boldsymbol{x}), \tag{6.42}$$

where $l(\rho)$ is the length of the partition (ρ). For $q \to 1$, $p_r(q;\boldsymbol{x}) \to m_r(\boldsymbol{x})$ while for $q \to 0$, $p_r(q;\boldsymbol{x}) \to (-1)^{r-1} m_{1^r}(\boldsymbol{x})$.

6.5 The q-deformation of HL Symmetric Functions

The q-deformation of the classical symmetric functions leads to a systematic development of the generalised *theory of* q-*deformed symmetric functions.* The q-analogue of the Hall-Littlewood symmetric functions will form the basis of the ring $\Lambda^q_{\mathbb{Q}}$ of the q-deformed symmetric functions. A q-analogue of complete symmetric functions can be defined as

$$h^q_\lambda = \sum_{|\lambda|=n} (z^q_\lambda)^{-1} p^q_\lambda, \tag{6.43}$$

where

$$z^q_\lambda = \prod [i]_q^{m_i} [m_i]_q!. \tag{6.44}$$

In chapter 2 we had mentioned that $P_\lambda(s,t)$ is the generalized form of the Hall-Littlewood symmetric function. Let us define a scalar product $\langle\ ,\ \rangle^{(q)}_{(s,t)}$ over $\mathcal{Q}_q(s,t)$ as follows:

$$\langle p_\lambda, p_\mu \rangle^{(q)}_{(s,t)} = \delta_{\lambda\mu} z^q_\lambda(s,t), \tag{6.45}$$

where

$$z^q_\lambda(s,t) = \prod_i [i]_q^{m_i} [m_i]_q! \prod_j^{l(\lambda)} \frac{(1 - s^{[\lambda_j]_q})}{(1 - t^{[\lambda_j]_q})}. \tag{6.46}$$

We call $P^q_\lambda(s,t)$, the q-deformation of the symmetric function $P_\lambda(s,t)$ and define

$$\mathcal{P}_q = \prod_{i,j} \left\{ \frac{(t x_i y_j; s)_\infty}{(x_i y_j; s)_\infty} \right\}_q,$$

where

$$(a;s)_\infty = \prod_{r=0}^{\infty} (1 - a s^r),$$

then we have

$$\mathcal{P}_q(x,y;s,t) = \sum_\lambda z^q_\lambda(s,t)^{-1} p_\lambda(x) p_\lambda(y). \tag{6.47}$$

Proof

We compute $\exp(\log \mathcal{P}_q)$;

$$\begin{aligned}
\log \mathcal{P}_q &= \sum_{i,j}\sum_{r=0}^{\infty}\left\{\log(1-x_iy_js^r)^{-1}-\log(1-tx_iy_js^r)^{-1}\right\}_q, \\
&= \sum_{i,j}\sum_{r=0}^{\infty}\sum_{n\geq 1}\frac{1}{[n]_q}(x_iy_js^r)^{[n]_q}\left(1-t^{[n]_q}\right), \\
&= \sum_{n\geq 1}\frac{1}{[n]_q}\frac{(1-t^{[n]_q})}{(1-s^{[n]_q})}p_n(x)p_n(y).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{P}_q &= \prod_{n\geq 1}\exp\left(\frac{1}{[n]_q}\frac{(1-t^{[n]_q})}{(1-s^{[n]_q})}p_n(x)p_n(y)\right), \\
&= \prod_{n\geq 1}\sum_{m_n=1}^{\infty}\frac{1}{[m_n]_q!}\left(\frac{1}{[n]_q}\frac{(1-t^{[n]_q})}{(1-s^{[n]_q})}p_n(x)p_n(y)\right)^{m_n},
\end{aligned}$$

in which the coefficient of $p_\lambda(x)p_\lambda(y)$ is seen to be $z_\lambda^q(s,t)^{-1}$. Here we have made use of the q-exponential function defined as

$$e_q^x = \sum_{n\geq 1}^{\infty}\frac{x^n}{[n]_q!}.$$

The above expression (6.47) is the most general definition of symmetric functions and all the symmetric functions (Hall-Littlewood, Schur's Q, Jack, zonal and Schur) are special cases of q-deformed symmetric functions. For $q=1$ and $s=0$, $P_\lambda^q(t)$ reduces to Hall-Littlewood symmetric functions and for $s=t^\alpha$, $q=1$ we get Jack symmetric functions, where α is an arbitrary parameter. For $q=1$ and $s=t$, $P_\lambda^q(s,t)$ reduces to S-functions. We can also have q-deformations of symmetric functions by setting $q\neq 0, \pm 1$ any arbitrary complex number. For example, $P_\lambda^q(0,t)$ or simply $P_\lambda^q(t)$ is the q-deformation of the Hall-Littlewood symmetric function which is of our major concern here.

Thus the scalar product $\langle\ ,\ \rangle_{(t)}^{(q)}$ over $\mathcal{Q}_q(t)$ is given as

$$\langle p_\lambda, p_\mu\rangle_{(t)}^{(q)} = \delta_{\lambda\mu}z_\lambda^q(t), \tag{6.48}$$

where

$$z_\lambda^q(t) = \prod_i [i]_q^{m_i}[m_i]_q!\prod_j^{l(\lambda)}\left(1-t^{[\lambda_j]_q}\right)^{-1}. \tag{6.49}$$

6.5.1 Duality and Orthogonality

Let us introduce another symmetric function $Q_\lambda^q(t)$ related with $P_\lambda^q(t)$ by a scalar $b_\lambda^q(t)$ as follows:

$$Q_\lambda^q(t) = b_\lambda^q(t)P_\lambda^q(t), \tag{6.50}$$

where

$$b_\lambda^q(t) = \prod_{i\geq 1} \phi^q_{m_i(\lambda)}(t), \qquad \phi_n^q(t) = \prod_{j\geq 1}^{n} \left(1 - t^{[j]_q}\right), \tag{6.51}$$

and m_i is the number of occurrences of i in λ. Then

$$\langle P_\lambda^q(t), Q_\mu^q(t)\rangle = \delta_{\lambda\mu},$$

i.e., $P_\lambda^q(t), Q_\lambda^q(t)$ are dual bases of $\Lambda_{\mathbb{Q}}^q$ for the scalar product $\langle\ ,\rangle$. It is easy to see that

$$\begin{aligned} Q_\lambda^q(t) &= \prod_{i<j} \left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q q_\lambda^q(t), \\ &= \prod_{i<j} \left(1 + (t^{[1]_q} - 1)\delta_{ij} + (t^{[2]_q} + t^{[1]_q})\delta_{ij}^2 + \cdots\right) q_\lambda^q(t), \end{aligned} \tag{6.52}$$

where $q_\lambda^q(t)$ are the projection of $Q_\lambda^q(t)$ defined as

$$\prod_i \left\{\frac{1-tx_iy}{1-x_iy}\right\}_q = \sum_{r=0}^{\infty} q_r^q(\boldsymbol{x};t)y^r,$$

and

$$q_\lambda^q(\boldsymbol{x};t) = \prod_i q_{\lambda_i}^q(\boldsymbol{x};t),$$

where y is a fictitious variable.

6.5.2 Recurrence Relations of Q-functions

The q-analogue of the recurrence relations obeyed by the Schur Q-functions as given in the section **2.7** can be defined as

$$\begin{aligned} Q^q_{\lambda_1\lambda_2\cdots\lambda_l} &= Q^q_{\lambda_1\lambda_2}Q^q_{\lambda_3\lambda_4\cdots\lambda_l} - Q^q_{\lambda_1\lambda_3}Q^q_{\lambda_2\lambda_4\cdots\lambda_l} \\ &\quad + \cdots + Q^q_{\lambda_1\lambda_l}Q^q_{\lambda_2\lambda_3\cdots\lambda_{l-1}} \qquad (l \text{ even}), \end{aligned}$$

and

$$\begin{aligned} Q^q_{\lambda_1\lambda_2\cdots\lambda_l} &= q^q_{\lambda_1}Q^q_{\lambda_2\lambda_3\cdots\lambda_l} - q^q_{\lambda_2}Q^q_{\lambda_1\lambda_3\cdots\lambda_l} \\ &\quad + \cdots + q^q_{\lambda_l}Q^q_{\lambda_2\lambda_3\cdots\lambda_{l-1}} \qquad (l \text{ odd}), \end{aligned}$$

and

$$Q^q_{\lambda_1\lambda_2} = q^q_{\lambda_1}q^q_{\lambda_2} - 2q^q_{\lambda_1+1}q^q_{\lambda_2-1} + \cdots + \left((-1)^{[\lambda_2]_q} + (-1)^{[\lambda_2-1]_q}\right) q^q_{\lambda_1+\lambda_2}.$$

The last relation is directly derived from Eq.(6.52). Also for $q_0 = 1$ and $q_{-s} = 0$ we have

$$Q^q_{-\lambda_r\lambda_r} = \left((-1)^{[\lambda_r]_q} + (-1)^{[\lambda_r-1]_q}\right) \quad \text{and} \quad Q^q_{\lambda_r,-\lambda_r} = 0.$$

6.6 q-Analogue of the Symmetric Group S_n

The q-deformation of the symmetric functions leads to the q-analogue of the characters of S_n.

The connection between the ordinary characters of S_n and S-functions can be given as

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda p_\mu, \tag{6.53}$$

where χ_μ^λ is the character of the irrep $\{\lambda\}$ for the class $\{\mu\}$ and p_μ are power sum symmetric functions.

The spin characters are related to Schur's Q-functions as follows:

$$Q_\lambda = 2^{[(l(\lambda)+l(\nu)+1)/2]} \sum_\nu z_\nu^{-1} \zeta_\nu^{[\Delta;\lambda]} p_\nu, \tag{6.54}$$

where $\zeta_\nu^{\Delta;\lambda}$ is the spin character for the class ν of odd cycles only and $[x]$ means the integer part of x.

We observe that for $s = t$, $P_\lambda^q(s,t)$ reduces to the q-deformed Schur function s_λ^q and for $s = 0$ & $t = -1$, $P_\lambda^q(s,t)$ reduces to the q-deformed Schur's Q-function . Hence we can make a q-analogue of the Equations (6.53) and (6.54) as follows:

$$s_\lambda^q = \sum_\mu (z_\mu^q)^{-1} \chi_\mu^\lambda(q) p_\mu, \tag{6.55}$$

and

$$Q_\lambda^q = 2^{[(l(\lambda)+l(\nu)+1)/2]} \sum_\nu (z_\nu^q)^{-1} \zeta_\nu^{[\Delta;\lambda]}(q) p_\nu. \tag{6.56}$$

6.6.1 q-Deformed Ordinary Characters

Using the techniques developed in the earlier sections we give the following algorithm for the calculation of the characters $\chi_\lambda^\mu(q)$.

Algorithm 5

1. Expand the S-function s_λ^q in terms of complete symmetric functions h_λ^q using the following:
$$s_\lambda^q = \prod_{i<j}(1-\delta_{ij})h_\lambda^q, \quad \text{where} \quad h_\lambda^q \equiv h_{\lambda_1}^q h_{\lambda_2}^q h_{\lambda_3}^q \cdots . \tag{6.57}$$

2. Expand h_n^q in terms of power sum symmetric functions p_μ as follows:
$$h_n^q = \sum_\mu (z_\mu^q)^{-1} p_\mu. \tag{6.58}$$

3. $\chi_\lambda^\mu(q)$ can be calculated by comparing the coefficients of p_μ on both sides of the Equation(6.55).

As an example, we give the q-dependent characters of S_4.

	1^4	21^2	22	31	4
4	1	1	1	1	1
31	$q+q^2+q^3$	q	-1	0	-1
22	q^2+q^4	$1-q$	$1+q$	-1	0
21^2	$q^3+q^4+q^5$	-1	$-q$	0	1
1^4	q^6	$-q$	q	1	-1

Table 6.1: $q-$dependent characters of S_4

It is easy to see that for $q = 1$ we get the usual characters of S_4.

6.6.2 q-Deformed Spin Characters

The spin characters of S_n are normally calculated by using the recurrence relations of the Q-functions along with the Eq.(6.54) [9]. In this section we will use the Eq.(6.56) and the q-analogue of the recurrence relations for the explicit calculations of the q-deformed spin characters.

Algorithm 6

1. Using Eq.(6.52), expand Q^q_λ in terms of q^q_r.

2. Write each q^q_r as
$$q^q_r = \sum_\rho (z^q_\rho)^{-1} 2^{l(\rho)} p_\rho,$$
where ρ is a partition of r.

3. Equate this to the expression (6.56) for Q^q_λ.

4. $\zeta^{\Delta;\lambda}_\nu(q)$ can be calculated by comparing the coefficients of p_ν on both sides of the equation 6.56).

Using the above algorithm we give the q-deformed spin characters of S_4 in table (6.2).

	1^4	21^2	22	31	4
$[\Delta;0]_+$	2	0	0	1	$\sqrt{2}$
$[\Delta;0]_-$	2	0	0	-1	$-\sqrt{2}$
$[\Delta;1]$	$2(q+q^2+q^3-1)$	0	0	-1	0

Table 6.2: $q-$dependent spin characters of S_4

It is important to note that the basic spin characters are independent of q.

6.6.3 Kostka-Foulkes Matrices

Kostka-Foulkes polynomials appear in the theory of the symmetric functions in the form of transition matrix [18, 20]. Originally the Kostka-Foulkes polynomials were defined in terms of Young tableaux but it has been shown that they are nothing but the inverse raising operators $\prod_{i<j}(1-t\delta_{ij})^{-1}$ matrix. Here we will give a $q-$analogue of *Kostka-Foulkes polynomials* $K_{\lambda\mu}(t)$ which appear in the following expansions:

$$s_\lambda^q(x) = \sum_\mu K_{\lambda\mu}^q(t) P_\mu^q(x;t), \tag{6.59}$$

and

$$Q_\lambda^q(x;t) = \sum_\mu K_{\lambda\mu}^q(t) S_\mu^q(x;t). \tag{6.60}$$

As an example, the $q-$Kostka-Foulkes polynomials $K_{\lambda\mu}^q(t)$ for $n = 2$, 3 & 4 are given in tables(6.3 - 6.5).

	2	1^2
2	1	$t^{[1]}$
1^2		1

Table 6.3: q-Kostka-Foulkes polynomials $K_{\lambda\mu}^q(t)$ for $n = 2$

	3	21	1^3
3	1	$t^{[1]}$	$t^{[3]}$
21		1	$t^{[1]} + t^{[2]}$
1^3			1

Table 6.4: $q-$Kostka-Foulkes polynomials $K_{\lambda\mu}^q(t)$ for $n = 3$

	4	31	22	21^2	1^4
4	1	$t^{[1]}$	$t^{[2]}$	$t^{[3]}$	$t^{[6]}$
31		1	$t^{[1]}$	$t^{[1]} + t^{[2]}$	$t^{[3]} + t^{[4]} + t^{[5]}$
22			1	$t^{[1]}$	$t^{[2]} + t^{[4]}$
21^2				1	$t^{[1]} + t^{[2]} + t^{[3]}$
1^4					1

Table 6.5: $q-$Kostka-Foulkes polynomials $K_{\lambda\mu}^q(t)$ for $n = 4$

6.7 Conclusion

A theory of q-deformed symmetric functions has been developed by introducing an arbitrary but fixed parameter q (where q is not a root of unity) in the original theory of symmetric functions. A technique similar to that of quantum groups has been followed. The ring Λ_Q^q of the q-deformed symmetric functions so achieved, finds many applications, one of which will be discussed in the subsequent chapter.

A logical consequence of this deformation is the q-analogue of the symmetric group S_n. Simple algorithms for the calculation of q-deformed ordinary and spin characters are given with illustrated example of S_4. Here we have used a particular type of q-deformation. The simple structure of Λ_Q^q makes it possible to achieve similar results, using any other type of q-deformation which will be shown in the next chapter.

Chapter 7

Vertex Operators

7.1 Introduction

The development of methods for constructing and studying integrable quantum models has recently led to new algebraic structures known as quantum groups [22] or more precisely quantum affine Lie algebras. Finding vertex operator representations of quantum affine algebras is a natural issue in the study of quantum groups. Besides, recent progress in conformal field theories has showed the important role played by vertex operator algebras in quantum field theories [36].

These developments have stimulated much activity in both mathematicians and physicists. In a recent paper [25] Frenkel and Jing have constructed the untwisted vertex representations of quantum affine algebras and more recently Jing [37] has developed the twisted q-vertex operators. Drinfeld's theorem of quantum affine algebras [22] plays the crucial role in such constructions. We find it very cumbersome.

In this chapter we will develop a very simple method for the construction of twisted and untwisted q-vertex operators by using a similar technique as adopted for the q-deformation of symmetric functions in the previous chapter.

First of all we will reconstruct the ring Λ_Q^q of q-deformed symmetric functions by using a different type of q-deformation then we will show that there exists an isomorphism between the ring Λ_Q^q and the space $\mathcal{V}_q$ of q-deformed vertex operators. These q-deformed vertex operators are nothing but the q-analogue of the untwisted vertex operators used in the description of affine Kac-Moody algebras [36]. This leads to a very simple way of constructing the twisted q-vertex operators.

7.2 The ring Λ_Q^q

In section (**6.5**), we gave the $q-$deformation of the function $P_\lambda(s,t)$ using the following definition of $q-$number

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} \tag{7.1}$$

It is possible to consistently define various types of $q-$deformations of symmetric functions such as in terms of the $q-$numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{7.2}$$

as used in the description of quantum groups [22]. Here we redefine the $q-$deformed scalar product $\langle , \rangle_{(s,t)}^{(q)}$ over $\mathcal{Q}_q(s,t)$ as

$$\langle p_\lambda, p_\mu \rangle_{(s,t)}^{(q)} = \delta_{\lambda\mu} z_\lambda^q(s,t), \tag{7.3}$$

where

$$z_\lambda^q(s,t) = \prod_i [i]_q^{m_i} [m_i]_q! \prod_j^{l(\lambda)} \frac{(1 - s^{[\lambda_j]_q})}{(1 - t^{[\lambda_j]_q})}, \quad \lambda = (1^{m_1} 2^{m_2} \ldots). \tag{7.4}$$

By the specialisation of (s,t,q), one can easily form the other results.

7.3 q-Analogue of Vertex Operators

Jing [21] has shown a relationship between vertex operators with a parameter t and the symmetric group S_n and its double covering group Γ_n. The parameter t plays a similar role in the description of vertex operators as it does in the theory of symmetric functions explained in the previous section i.e. the vertex operators with $t = 0$ correspond to S-functions and those with $t = -1$ correspond to Schur's Q-functions. Here we shall give a $q-$analogue of vertex operators and will show a 1-1 correspondence between the space of $q-$deformed vertex operators $\mathcal{V}_q$ and the ring of $q-$deformed symmetric functions $\mathcal{Q}_q(t)$.

Vertex operators are defined with the help of infinite dimensional Heisenberg algebras.

We shall define a $q-$analogue of a Heisenberg algebra $\mathcal{H}$ as

Definition 18 *The $q-$Heisenberg algebra $\mathcal{H}_q$ is generated by a and ζ_n, $n \in \mathcal{Z}/0$, and satisfies the following relations:*

$$[\zeta_m, \zeta_n]_q = \frac{[m]_q}{1 - t^{[m]_q}} \delta_{m+n,0} a, \quad [\zeta_m, a]_q = 0, \tag{7.5}$$

where t is a parameter.

As usual $S(\mathcal{H}_q^-)$ is the symmetric algebra generated by ζ_{-n}, $n \in \mathbf{N}$. ζ_{-n} is regarded as a multiplication operator and ζ_n as an annihilation operator on $S(\mathcal{H}_q^-)$. As an example,

$$\zeta_n \zeta_{-n}.1 = \frac{[n]_q}{1 - t^{[n]_q}} \quad n \in N,$$

where a is considered as an identity operator.

Now we can define the q-analogue of a simplified form of vertex operators on the space $S(\mathcal{H}_q^-)$ as follows:

$$\begin{aligned} V(x) &= \exp\left\{\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x^n\right\}\exp\left\{-\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_n x^{-n}\right\}, \\ &= \sum_{n\in\mathbf{Z}} V_n x^{-n}, \end{aligned} \tag{7.6}$$

$$\begin{aligned} V^*(x) &= \exp\left\{-\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x^n\right\}\exp\left\{\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_n x^{-n}\right\}, \\ &= \sum_{n\in\mathbf{Z}} V_n^* x^{n}. \end{aligned} \tag{7.7}$$

We define a hermitian structure $\langle , \rangle$ in the space $S(\mathcal{H}_q^-)$

$$\langle\zeta_{-n},\zeta_{-n}\rangle = \frac{[n]_q}{1-t^{[n]_q}},$$

or in general

$$\langle\zeta_{-\lambda},\zeta_{-\mu}\rangle = z_\lambda^q(t)\delta_{\lambda\mu},$$

where $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_l)$ and $\mu = (\mu_1,\mu_2,\ldots,\mu_k)$ are partitions and $z_\lambda^q(t)$ is defined as

$$z_\lambda^q(t) = \prod [i]_q^{m_i}[m_i]_q! \prod_{j\geq 1}\left(1-t^{[\lambda_j]_q}\right)^{-1}, \quad \lambda = (1^{m_1}2^{m_2}\ldots). \tag{7.8}$$

A polynomial function in ζ_{-n} can be defined as follows:

$$\exp\left(\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x^n\right) = \sum_{n\geq 0} R_n^q(t)x^n.$$

Hence

$$R_n^q(t) = \sum_{|\lambda|=n}\left(z_\lambda^q(t)\right)^{-1}\zeta_{-\lambda}, \quad \zeta_{-\lambda} = \zeta_{-\lambda_1}\zeta_{-\lambda_2}\cdots\zeta_{-\lambda_l}. \tag{7.9}$$

The normal ordering product is used when the annihilation operator has to be moved to the right of the product [46], as shown below.

$$: V(x)V(y) := \exp\left\{\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}(x^n+y^n)\right\}\exp\left\{-\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_n(x^{-n}+y^{-n})\right\},$$

and

$$V(x)V(y) =: V(x)V(y) : \left\{\frac{x-y}{x-ty}\right\}_q,$$

where the subscript q indicates that the factor $\left\{\frac{x-y}{x-ty}\right\}_q$ is a formal series in $\frac{y}{x}$ with the powers of t being q-numbers.

Using the $q-$analogue of Young raising operators we give a $q-$analogue of Jing's proposition(2.17) [21] as follows:

Theorem 7 *For a partition $\lambda = (\lambda_1\lambda_2\ldots\lambda_l)$ the element $V_{-\lambda}.1$ can be expressed as*

$$V_{-\lambda}.1 = \prod_{i<j}\left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q R^q_\lambda(t)$$

where δ_{ij} is Young's raising operator whose action is defined as

$$\delta_{ij}R^q_{(\lambda_1\cdots\lambda_i\cdots\lambda_j\cdots)} = R^q_{(\lambda_1\cdots\lambda_i+1\cdots\lambda_j-1\cdots)},$$

and the subscript q in $\{\ \}_q$ means the powers of t are q-numbers.

Proof

The action of the components of the vertex operators $V(x)$ as defined in Eq. (7.6), can be shown as

$$V_{-n}.1 = \frac{1}{2\pi i}\int_c \exp\left(\sum_{m\geq 1}\frac{1-t^{[m]_q}}{[m]_q}\zeta_{-m}x^m\right)x^{-n}\frac{dx}{x},$$

where the subscript c is for the contour integral.

Then it is easy to see the trivial result

$$V_{-n}.1 = R^q_n(t).$$

For the rest, let us use the contour integral approach. For any partition $\lambda = \lambda_1\cdots\lambda_l$,

$$\begin{aligned} V_{-\lambda}.1 &= \underbrace{\int\cdots\int}_{l} V(x_1)\cdots V(x_l).1x^{-\lambda}\frac{dx_1}{x_1}\cdots\frac{dx_l}{x_l} \\ &= \frac{1}{(2\pi i)^l}\int \exp\left(\sum_{i=1,n\geq 1}^{l}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}x_i^n\right)\prod_{1\leq i<j\leq l}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q x^{-\lambda}\frac{dx}{x}, \end{aligned}$$

where the term $\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q$, comes from the normal ordering of the creation and annihilation operators. Using the definition of $R^q_n(t)$, we can write the following.

$$V_{-\lambda}.1 = \frac{1}{(2\pi i)^l}\int\sum_{n\in N^l}R^q_n(t)\prod_{i<j}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q x^{-\lambda+n}\frac{dx}{x}.$$

Expanding the formal series $\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q$ for $i=1$ we get

$$V_{-\lambda}.1 = \frac{1}{(2\pi i)^l}\int\sum_{n\in N^l}R^q_n(t)\prod_{j=2}^{l}\left(1+(t^{[1]_q}-1)\frac{x_j}{x_1}+(t^{[2]_q}-t^{[1]_q})\frac{x_j^2}{x_1^2}+\cdots\right)$$

$$
\begin{aligned}
&\prod_{2\le i<j\le l}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q x^{-\lambda+n}\frac{dx}{x},\\
= \;& \frac{1}{(2\pi i)^{l-1}}\prod_{j=2}^{l}\left(1+(t^{[1]_q}-1)\delta_{1j}+(t^{[2]_q}-t^{[1]_q})\delta_{1j}^2+\cdots\right)\int\sum_{\tilde{n}}R^q_{\lambda_1}(t)R^q_{\tilde{n}}(t)\\
&\prod_{2\le i<j\le l}\left\{\frac{x_i-x_j}{x_i-tx_j}\right\}_q \tilde{x}^{-\tilde{\lambda}+\tilde{n}}\frac{d\tilde{x}}{\tilde{x}},\\
= \;& \prod_{i<j}\left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q R^q_\lambda(t),
\end{aligned}
$$

where $\tilde{\lambda}=\lambda_2,\lambda_3,\cdots\lambda_l$, $\tilde{x}=x_2\cdots x_l$, $\frac{dx}{x}=\frac{dx_1}{x_1}\cdots\frac{dx_l}{x_l}$ and $R^q_\lambda(t)=R^q_{\lambda_1}(t)R^q_{\lambda_2}(t)\cdots R^q_{\lambda_l}(t)$. The orthogonality of the $q-$analogue of vertex operators can be described as

Theorem 8 *For two partitions $\lambda=(\lambda_1,\lambda_2,\ldots\lambda_k)$ and $\mu=(\mu_1,\mu_2,\ldots\mu_l)$*

$$\langle V_{-\lambda}.1, V_{-\mu}.1\rangle_q = b^q_\lambda(t)\delta_{\lambda\mu}, \tag{7.10}$$

where

$$b^q_\lambda(t)=\prod_{i\ge1}\phi^q_{m_i(\lambda)}(t),\qquad \phi^q_n(t)=\prod_{j\ge1}^{n}\left(1-t^{[j]_q}\right), \tag{7.11}$$

and m_i is the number of occurrences of i in λ.

In order to prove this we will give q-analogues of some of Jing's results [21].

Lemma 1 *For $m,n\in N$, we have*

$$V^*_{-n}V_{-m}.1=\delta_{m,n}(1-t^{[1]_q}).$$

The proof of this lemma is straight forward, by using the properties of the components of vertex operators.

Proposition 1 *Let $\lambda=(1^{m_1},2^{m_2},\cdots)$ and $\tilde{\lambda}=(1^{m_1-1},2^{m_2},\cdots)$ then we have*

$$V^*_{-n}V_{-\lambda}.1=\delta_{n,\lambda_1}\left(1-t^{[m_1]_q}\right)V_{-\tilde{\lambda}}.1$$

The above lemma and the inductive assumptions prove the above proposition.

Now the orthogonality of the q-deformed vertex operators can be proved as follows.

For two partitions λ and μ, such that $|\lambda|=|\mu|$ we have

$$
\begin{aligned}
\langle V_{-\lambda}.1, V_{-\mu}.1\rangle_q &= \langle V_{-\tilde{\lambda}}.1, V^*_{-\lambda_1}V_{-\mu}.1\rangle_q,\\
&= \langle V_{-\tilde{\lambda}}.1, \delta_{\lambda_1,\mu_1}\left(1-t^{[m_1(\mu)]_q}\right)V_{-\tilde{\mu}}.1\rangle_q,\\
&= \delta_{\lambda_1,\mu_1}\left(1-t^{[m_1(\mu)]_q}\right)\langle V_{-\tilde{\lambda}}.1, V_{-\tilde{\mu}}.1\rangle_q.
\end{aligned}
$$

By repeating the above we get

$$\langle V_{-\lambda}.1, V_{-\mu}.1\rangle_q = b^q_\lambda(t)\delta_{\lambda\mu},$$

which is the desired result.

Comparing the inner product $\langle\zeta_{-\lambda},\zeta_{-\mu}\rangle=z^q_\lambda(t)\delta_{\lambda\mu}$, and the Eq. (6.48), we can define a mapping from $\mathcal{V}_q$ to $\Lambda\otimes_{\mathcal{Z}}\mathcal{Q}_q(t)$ as follows:

Definition 19 *The mapping* $\rho : \mathcal{V}_q \to \Lambda \otimes_Z \mathcal{Q}_q(t)$ *for a partition* $\lambda = (1^{m_1} 2^{m_2} ... l^{m_l})$ *is given as*

$$\rho\left(\zeta_{-\lambda}\right) = \rho\left(\zeta_{-1}^{m_1} \zeta_{-2}^{m_2} ... \zeta_{-l}^{m_l}\right) = p_1^{m_1} p_2^{m_2} ... p_l^{m_l} = p_\lambda.$$

This immediately gives

$$\rho\left(V_{-\lambda}.1\right) = \rho\left(V_{-\lambda_1} V_{-\lambda_2} ... V_{-\lambda_l}.1\right) = \prod_{i<j} \left\{ \frac{1 - \delta_{ij}}{1 - t\delta_{ij}} \right\}_q q_\lambda^q. \tag{7.12}$$

Comparing (7.10) and the identity (see section **6.5**)

$$\langle Q_\lambda^q(t), Q_\mu^q(t) \rangle = b_\lambda^q(t) \delta_{\lambda\mu},$$

we conclude that for a general value of t the map $\rho : \mathcal{V}_q \to \Lambda \otimes_Z \mathcal{Q}_q(t)$ takes the form

$$\rho\left(V_{-\lambda_1} V_{-\lambda_2} ... V_{-\lambda_l}(t).1\right) = Q_\lambda^q(t). \tag{7.13}$$

The specializations $t = 0\,, -1$ give the following results:

$$\rho\left(V_{-\lambda_1} V_{-\lambda_2} ... V_{-\lambda_l}(-1).1\right) = Q_\lambda^q(-1), \tag{7.14}$$

and

$$\rho\left(V_{-\lambda_1} V_{-\lambda_2} ... V_{-\lambda_l}(0).1\right) = s_\lambda^q, \tag{7.15}$$

where $Q_\lambda^q(-1)$ are the q-deformed Schur Q-functions and s_λ^q are the q-deformed S-functions.

7.4 Construction of Untwisted q-Vertex Operators

In previous section we have worked out the q-analogue of the vertex operators. On the basis of Eqs. (7.6) and (7.7) we define the untwisted q-vertex operators in normal ordered form as follows:

$$\begin{aligned} U(z) &= \exp\left\{ \sum_{n\geq 1} \frac{1 - t^{[n]}}{[n]} \zeta_{-n} z^n \right\} \exp\left\{ - \sum_{n\geq 1} \frac{1 - t^{[n]}}{[n]} \zeta_n z^{-n} \right\} e^\zeta z^{\zeta_{(0)}+1}, \\ &= \sum_{n\in \mathbf{Z}} U_n z^{-n}, \\ &= \, : U(z) :, \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} U^*(z) &= \exp\left\{ - \sum_{n\geq 1} \frac{1 - t^{[n]}}{[n]} \zeta_{-n} z^n \right\} \exp\left\{ \sum_{n\geq 1} \frac{1 - t^{[n]}}{[n]} \zeta_n z^{-n} \right\} e^\zeta z^{-(\zeta_{(0)}+1)}, \\ &= \sum_{n\in \mathbf{Z}} U_n^* z^n, \\ &= \, : U^*(z) :, \end{aligned} \tag{7.17}$$

where z is a non-zero complex number and the action of $\zeta_{(0)}$ is defined as

$$\zeta_{(0)} e^\eta = \langle \eta, \zeta \rangle e^\eta \qquad \eta, \zeta \in S(\mathcal{H}_q^-).$$

The factors $e^{\zeta}z^{-(\zeta_{(0)}+1)}$ and $e^{\zeta}z^{\zeta_{(0)}+1}$ arise from the commutation of annihilation operators as they are transferred to the right in accordance with normal ordering. For $t \to 0$ and $q \to 1$ the above expressions take a similar form to the vertex operators used in dual resonance theory [42].

There is another way of developing the untwisted q-vertex operators. Consider a finitely generated free abelian group L and define a nonsingular symmetric $\mathcal{Z}$- bilinear form $\langle , \rangle$ on L such that

$$\langle \zeta, \zeta \rangle \in 2\mathcal{Z} \quad \text{for} \quad \zeta \in L.$$

Define the function

$$C : L \times L \to \mathcal{F},$$
$$(\zeta, \eta) \mapsto (-1)^{\langle \zeta, \eta \rangle} \omega^{\langle m\zeta, \eta \rangle} = \prod (-\omega^m)^{\langle \zeta, \eta \rangle},$$

where ω is the kth primitive root of unity and $m \in \mathcal{Z}/k\mathcal{Z}$. Then the commutator map C is bilinear into the abelian group $\mathcal{F}$ such that

$$\begin{aligned} C(\zeta + \eta, \theta) &= C(\zeta, \theta)C(\eta, \theta), \\ C(\zeta, \eta + \theta) &= C(\zeta, \eta)C(\zeta, \theta), \end{aligned} \tag{7.18}$$

and

$$C(\zeta, \zeta) = 1, \tag{7.19}$$

for $\zeta, \eta, \theta \in L$.

Let $\omega_0 = (-1)^k \omega$. In view of the Eqs. (7.18) and (7.19) there is a unique central extension

$$1 \to \langle \omega_0 \rangle \to \hat{L} \xrightarrow{-} L \to 1, \tag{7.20}$$

of L by the cyclic group generated by ω_0 with commutator map C such that

$$aba^{-1}b^{-1} = C(\bar{a}, \bar{b}) \quad \text{for} \quad a, b \in \hat{L}.$$

We fix $a \in \hat{L}$ such that $\bar{a} = \zeta$. This construction gives us the following form of the q-vertex operators.

$$\begin{aligned} \mathcal{X}(z) &= \exp\left\{ \sum_{n \geq 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_{-n} z^n \right\} \exp\left\{ - \sum_{n \geq 1} \frac{1 - t^{[n]_q}}{[n]_q} \zeta_n z^{-n} \right\} a z^{\zeta_{(0)}+1}, \\ &= \sum_{n \in \mathbf{Z}} \mathcal{X}_n z^n, \end{aligned} \tag{7.21}$$

where the $a \in \hat{L}$. For a special case of

$$\langle \omega_0 \rangle \equiv \langle \pm 1 \rangle,$$

we get

$$aba^{-1}b^{-1} = (-1)^{\langle \bar{a}, \bar{b} \rangle} \quad \text{for} \quad a, b \in \hat{L},$$

and the untwisted q-vertex operators take the following form.

$$\begin{aligned}\mathcal{X}^{\pm}(z) &= \exp\left\{\pm\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{-n}z^{n}\right\}\exp\left\{\mp\sum_{n\geq 1}\frac{1-t^{[n]_q}}{[n]_q}\zeta_{n}z^{-n}\right\}a^{\pm 1}z^{\pm\zeta_{(0)}+1},\\ &= \sum_{n\in\mathbf{Z}}\mathcal{X}^{\pm}_{n}z^{n}. \end{aligned} \tag{7.22}$$

For $t=0$ the above expression is similar to the one given by Frenkel and Jing [25] in the description of quantum affine algebras, except that they have used a different definition of q-number. The Eq. (7.22) is the most general expression for the vertex operators of *untwisted* type. With the various specializations of q and t one can derive all the vertex operators discussed earlier.

7.5 Construction of Twisted q-Vertex Operators

The *twisted* vertex operators are obtained by the action of an automorphism of the group Q. Closely following the terminology and notations used in [24] and using the results of the previous section, we define the following.

1. Q is a finitely generated free abelian group.
2. $\langle,\rangle$ is a nonsingular symmetric $\mathcal{Z}$- bilinear form on Q such that
$$\langle\zeta,\zeta\rangle\in 2\mathcal{Z}\quad\text{for}\quad\zeta\in Q.$$
3. σ is an automorphism of Q such that
$$\langle\sigma\zeta,\sigma\eta\rangle=\langle\zeta,\eta\rangle\quad\text{for}\quad\zeta,\eta\in Q.$$
4. m is a positive integer such that $\sigma^m=1$.
5. $$\left\langle\sum_{p\in\mathcal{Z}/m\mathcal{Z}}\sigma^{p}\zeta,\zeta\right\rangle\in 2\mathcal{Z}\quad\text{for}\quad\zeta\in Q.$$

Considering the action of the automorphism σ we redefine the commutator map C as follows.

$$C:Q\times Q\to\mathcal{F},$$
$$(\zeta,\eta)\mapsto(-1)^{\left\langle\sum_{p\in\mathcal{Z}/m\mathcal{Z}}\sigma^{p}\zeta,\eta\right\rangle}\omega^{\left\langle\sum_{p\in\mathcal{Z}/m\mathcal{Z}}p\sigma^{p}\zeta,\eta\right\rangle}=\prod_{p\in\mathcal{Z}/m\mathcal{Z}}(-\omega^{p})^{\langle\sigma^{p}\zeta,\eta\rangle}.$$

Along with the Eqs. (7.18) and (7.19) we include the following:

$$C(\sigma\zeta,\sigma\eta)=C(\zeta,\eta)\quad\text{for}\quad\zeta,\eta\in Q,\tag{7.23}$$

and

$$C(\zeta,\eta)=C(\eta,\zeta)^{-1}\quad\text{for}\quad\zeta,\eta\in Q.\tag{7.24}$$

Then the central extension of Q by the cyclic group generated by ω_0 with the commutator map C is

$$1 \to \langle\omega_0\rangle \to \hat{Q} \bar{\to} Q \to 1, \tag{7.25}$$

such that

$$aba^{-1}b^{-1} = C(\bar{a}, \bar{b}) \quad \text{for} \quad a, b \in \hat{Q}.$$

The automorphism σ can be *extended* to an automorphism $\hat{\sigma}$ of the extension $\hat{Q}$ of Q such that

$$(\hat{\sigma}a)^- = \sigma\bar{a} \quad \forall\, a \in \hat{Q},$$

and

$$\hat{\sigma}a = a\omega^{-\sum_{p\in Z/mZ} \sigma^p\bar{a}(0) - \langle\sigma^p\bar{a},\bar{a}\rangle/2}.$$

Now the twisted q-vertex operators can be defined as

$$\mathcal{X}(z) = \mathcal{E}_-(\zeta, z)\mathcal{E}_+(\zeta, z)az^{-\sum_{p\in Z/mZ} \sigma^p\bar{a}(0) - \langle\sigma^p\bar{a},\bar{a}\rangle/2}, \tag{7.26}$$

where

$$\mathcal{E}_\pm = \exp\left\{\sum_{\pm n\geq 1} \frac{1 - t^{[n]_q}}{[n]_q}\zeta_{-n}z^n\right\}.$$

Again for the special case of

$$\langle\omega_0\rangle \equiv \langle\pm 1\rangle,$$

we get

$$aba^{-1}b^{-1} = (-1)^{\langle\bar{a},\bar{b}\rangle} \quad \text{for} \quad a, b \in \hat{Q},$$

and the twisted q-vertex operators take the form

$$\begin{aligned} \mathcal{X}^\pm(z) &= \mathcal{E}_-^\pm(\zeta, z)\mathcal{E}_+^\pm(\zeta, z)a^{\pm 1}z^{\pm\sum_{p\in Z/mZ} \sigma^p\bar{a}(0) - \langle\sigma^p\bar{a},\bar{a}\rangle/2}, \\ &= \sum_{n\in\mathbf{Z}} \mathcal{X}_n^\pm z^n, \end{aligned} \tag{7.27}$$

where

$$\mathcal{E}_-^\pm = \exp\left\{\pm\sum_{n\geq 1} \frac{1 - t^{[n]_q}}{[n]_q}\zeta_{-n}z^n\right\},$$

and

$$\mathcal{E}_+^\pm = \exp\left\{\mp\sum_{n\geq 1} \frac{1 - t^{[n]_q}}{[n]_q}\zeta_n z^n\right\}.$$

For the specialisation $t = 0$ the Eq. (7.27) gives the similar result as reported in [37] except that the definition of q-number is different. Also for $q = 1$ and $t = 0$ the above result is similar to the case studied by Lepowsky [24]. We find that the expression (7.27) is a very general form of vertex operators. Various specializations of t, q and σ give the desired results. For example, in the case of identity automorphism $\sigma = 1$ we get the *untwisted* q-vertex operators and the Eq. (7.27) reduces to the Eq. (7.22).

7.6 Conclusion

A q-analogue of the Heisenberg algebra is defined. This leads to the construction of q-vertex operators with a parameter t similar to the theory of symmetric functions. An isomorphism from the space of q-vertex operators to the ring Λ_Q^q of the q-deformed vertex operators is defined explicitly. This isomorphism is valid for a general value of t and as well as the specialised values such that $t = 0$ and $t = -1$ in which case we get S-functions and Schur Q-functions. Using these results a very simple technique for the construction of *twisted* and *untwisted* q-vertex operators is developed. This approach is more simple and straight forward than any other technique. The final result is a very general form of the vertex operators and by the specialisations of various parameters, the results can be verified.

Chapter 8

Concluding Remarks

This thesis deals with several aspects of the theory of symmetric functions with regard to its applications in physical problems.

The symmetric functions find their applications in the atomic and molecular physics through the symmetric group S_n. The role of S-functions in the ordinary character theory of S_n is well-known. A more general form of symmetric functions, called Schur Q-functions, play a similar role in the theory of projective representations of S_n. Exploring the properties of Young raising operators and the Q-functions we completed the branching rule $O_n \to S_n$ for ordinary and spin characters. A furthur study of Q-functions and the combinatorial theory of shifted tableaux lead to the development of some important algorithms for the calculation of Kronecker products of Q-functions.The main drawback of the combinatorial theory of shifted tableaux is that it generates a rather large number of *dead* tableaux during these calculations. The problem is so severe that even a powerful computer cannot handle it. Carefully studying the properties of shifted skew tableaux, we managed to get rid of all the dead tableaux arising in the calculation of Kronecker products of Q-functions.

These developments logically lead to the calculation of the Kronecker products of the spin irreps of S_n. The algorithms so developed are easily *computable*, that is, the corresponding computer algorithms are easy to write and simple in working. This is done by incorporating these algorithms in the existing program SCHUR. The typical examples of the output of SCHUR shows the power of these algorithms.

Another important application of symmetric functions is found in conformal field theory. It has been recently discovered and has drawn much attention from both mathematicians and physicists. The theory of symmetric functions plays an important role in study of algebraic structures of quantum mechanical models. The study of quantum groups has lead to several types of q-deformations of these algebraic structures. The q- deformation of vertex operators has also been introduced since the vertex operator algebras are very powerful mathematical tool in the study of quantum affine Lie algebras. Unfortunately there did not exist any theory of q-deformed symmetric functions. In this thesis we present a parallel theory of q-deformed symmetric functions. This has been achieved by introducing an arbitrary but fixed parameter q (where q is not a root of unity) in the original theory of symmetric functions. A technique similar to the quantum groups has been followed. A logical consequence of this deformation is the q-analogue of the symmetric group S_n. Simple algorithms for the calculation of q-deformed ordinary and spin characters are given with illustrated examples for S_4.

This formalism leads to a very simple technique of the construction of twisted and untwisted q-vertex operators. At first, a q-analogue of vertex operators is defined with the help of a q-deformed Heisenberg algebra, then an isomorphism from the space of q-vertex operators to the ring Λ_Q^q of the q-deformed vertex operators is defined explicitly. Using these results a very simple technique for the construction of *twisted* and *untwisted* q-vertex operators is developed. This approach is more simple and straight forward than any other technique. The final result is a very general form of the vertex operators and by the specialisations of various parameters, the results can be verified.

Using the techniques developed for the calculation of inner products of Q-functions and the properties of shifted tableaux explored, it is possible to find a simple technique for the calculation of Q-functions *plethysm*, similar to that of S-functions.

The q-deformation of symmetric functions and the development of q-vertex operators give a new insight of the applications of the symmetric functions in physics and specially in quantum field theory. Using the twisted or untwisted q-vertex operators along with the isomorphism $\rho : \mathcal{V}_q \to \Lambda_Q^q$, one can construct the q-deformed symmetric function representations of quantum affine algebras.

Bibliography

[1] Schur I 1911 *J. Reine Angew. Math.* **139** 155

[2] Hall P 1957 *Proc. Fourth Canadian Math. Congress, Banff* p 147

[3] Littlewood D E 1961 *Proc. London Math. Soc.* (3) **11** 485

[4] Sagan B E 1987 *J. Combin. Theory. Ser. A* **45** 62-103

[5] Worley D R 1989 Ph.D. Thesis, M. I. T.

[6] Murnaghan F D 1937 *Am. J. Math.* **59** 739

[7] Murnaghan F D 1938 *Am. J. Math.* **60** 761

[8] Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 327-348

[9] Morris A O 1962 *Proc. Lond. Math. Soc.* **12** 55-76

[10] Salam M A and Wybourne B G 1989 *J. Phys. A: Math. Gen.* **22** 3771-3778

[11] Littlewood D E 1950 *The Theory of Group Characters* (Oxford University Press) 2nd edn

[12] Morris A O 1962 *J. London Math. Soc.* **37** 445-455

[13] Stembridge J R 1989 *Advances in Mathematics* **74** 87-134

[14] Stembridge J R *Private communication*

[15] King R C 1975 *J. Phys. A: Math. Gen.* **8** 429

[16] Thomas G P 1981 *Can. J. Math.* **34** 49

[17] Salam M A and Wybourne B G 1990 *J. Math. Phys.* **31** 1310-1314

[18] Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford University Press)

[19] Nimmo J J C 1990 *J.Phys.A:Math.Gen.* **23** 751-760

[20] Macdonald I G 1988 *A New Class of Symmetric Functions*, Queen Mary College, Preprint

[21] Jing N H 1990 *Twisted Vertex Representations of Quantum Affine Algebras*, Institute for Advanced Study, Preprint

[22] Drinfeld V G 1988 *Soviet Math. Dokl.* **36**, 21 2-216

[23] Hamermesh M 1962 *Group theory and its application to physical problems* Addison-Wesley Publishing Company.

[24] Lepowsky J 1985 *Proc. Natl. Acad. Sci. USA* **82** 8295-8299.

[25] Frenkel I B and Jing N H 1988 *Proc. Natl. Acad. Sci. USA* **85** 9373-9377.

[26] Schur I 1901 *Inaugural Dissertation* Berlin.

[27] Emmerson G E 1977 *John Scott Russell* John Murray Publishers.

[28] Stone M 1990 *Vertex Operators in Quantum Hall Effect* Preprint (University of Illinois at Urbana Champaign) USA.

[29] Kadomtsev V V and Petviashvili V I 1970 *Sov. Phys. Dokl.* **15** 539.

[30] Sato M 1981 *RIMS Kokyuroku* **472** 62.

[31] Jimbo M and Miwa T 1983 *RIMS Kyoto University* **19** 943.

[32] Date E, Jimbo M, Kashiwara M and Miwa T 1982 *Physica* **4D** 343-365.

[33] Salam M A and Wybourne B G 1990 *A note on Young raising operators and Schur Q-functions* Preprint, University of Canterbury, New Zealand.

[34] Salam M A and Wybourne B G 1991 *J. Phys. A: Math. Gen.* **24** L317-21.

[35] Salam M A and Wybourne B G 1991 *Symmetric functions and vertex operators* Preprint, University of Canterbury, New Zealand.

[36] Goddard P and Olive D 1986 *Int. J. Mod. Phys A* **1** 303-414.

[37] Jing N H 1990 *Twisted vertex representations of quantum affine algebras* Preprint, Inst. Adv. Study, USA.

[38] Jimbo M 1985 *Lett. Math. Phys.* **10** 63-69.

[39] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873-8.

[40] King R C and Wybourne B G 1991 *J. Phys. A: Math. Gen.* **23** L1193-7.

[41] Koornwinder T H 1989 *Proc. Kon. Ned. Akad. van Wetensch* **A 92** 97-117.

[42] Schwartz J H 1973 *Phys. Rep.* **8** 269-335.

[43] MacDonald A H 1984 *Phys. Rev. B* **29** 6563.

[44] MacDonald A H 1984 *Phys. Rev. B* **30** 4392.

[45] Woronowicz S L 1987 *Commn. Math. Phys.* **111** 613-665.

[46] Frenkel I B, Lepowsky J and Meurman A 1988 *Vertex operator algebras and the monster* Academic Press.

Acknowledgements

I would like to thank the persons and organisations who supported me during the work for this thesis.

I greatly value and appreciate the patience, excellent guidance and the moral support of my teacher and supervisor Prof. B. G. Wybourne. His continuous encouragement and personal interest provided a strong foundation for the skills and techniques I learnt in the course of this thesis. I have not only learnt physics and mathematics from him but the values of life as well.

I want to thank Prof. G. E. Stedman for his encouragement and support during the absence of Prof. B. G. Wybourne and for the valuable comments on this thesis which I found a great help. A special thanks to Dr. B. G. Searle for his expert suggestions during the programming of some parts of this thesis. I also appreciate some very helpful discussions with my fellow student Mr. H. J. Ross.

I would like to thank Mary Kinnaird, Gill Evans and Beverly Bristowe for their love and care during my stay in the physics department. Thanks are also due to Mr. C. A. Blockley and Mr. D. J. Gunn for their friendships which I greatly value.

I would also like to thank the physics department for the financial support in form of teaching assistantship and to the University of Canterbury for the Roper scholarship without which this work would not have done.

Finally, I want to acknowledge the support and encouragement of my family.